Seminar 3

Seemingly Unrelated Regression (SUR) II

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Problem #5 from Baltagi, Chapter 10

(a) Show that var($\hat{\beta}_{1,\text{OLS}}$) = $\frac{\sigma_{11}}{m_{x_1 x_1}}$ and var($\hat{\beta}_{2,\text{OLS}}$) = $\frac{\sigma_{22}}{m_{x_2 x_2}}$, where $m_{x_i x_j} = \sum_{t=1}^{T} \left(X_{\text{it}} - \overline{X_i}\right)\left(X_{\text{jt}} - \overline{X_j}\right)$ for $i,\ j = 1,\ 2$.

Solution:

$\hat{\beta}_{1,\text{OLS}} = \left(X_1^T A X_1\right)^{-1} X_1^T A Y_1,\ A = \left(I_T - \frac{i\,i^T}{T}\right),\ X_1^T A X_1 = \sum_{t=1}^{T} \left(X_{1\,t} - \overline{X_1}\right)\left(X_{1\,t} - \overline{X_1}\right) = m_{x_1 x_1}$

var($\hat{\beta}_{1,\text{OLS}}$) = $\sigma_{11}\left(X_1^T A X_1\right)^{-1} = \frac{\sigma_{11}}{m_{x_1 x_1}}$. Similarly we proof that var($\hat{\beta}_{2,\text{OLS}}$) = $\frac{\sigma_{22}}{m_{x_2 x_2}}$

(b) $\text{var}\begin{pmatrix}\hat{\beta}_{1,\text{GLS}}\\ \hat{\beta}_{2,\text{GLS}}\end{pmatrix} = \left(\sigma_{11}\,\sigma_{22} - \sigma_{12}^2\right)\begin{pmatrix}\sigma_{22}\, m_{x_1 x_1} & -\sigma_{12}\, m_{x_1 x_2}\\ -\sigma_{12}\, m_{x_1 x_2} & \sigma_{11}\, m_{x_2 x_2}\end{pmatrix}^{-1}$. Deduce that $\text{var}(\hat{\beta}_{1,\text{GLS}}) = \left(\sigma_{11}\,\sigma_{22} - \sigma_{12}^2\right)\sigma_{11}\, m_{x_2 x_2} \big/ \left(\sigma_{11}\,\sigma_{22}\, m_{x_2 x_2}\, m_{x_1 x_1} - \sigma_{12}^2\, m_{x_1 x_2}^2\right)$ and $\text{var}(\hat{\beta}_{2,\text{GLS}}) = \left(\sigma_{11}\,\sigma_{22} - \sigma_{12}^2\right)\sigma_{22}\, m_{x_1 x_1} \big/ \left(\sigma_{11}\,\sigma_{22}\, m_{x_1 x_1}\, m_{x_2 x_2} - \sigma_{12}^2\, m_{x_1 x_2}^2\right)$

Solution:

$$\begin{pmatrix}\hat{\beta}_{1,\text{GLS}}\\ \hat{\beta}_{2,\text{GLS}}\end{pmatrix} = \left(\begin{pmatrix}X_1^{\text{T}}\text{A} & 0\\ 0 & X_2^{\text{T}}\text{A}\end{pmatrix}\begin{pmatrix}\sigma^{11}\otimes\text{I} & \sigma^{12}\otimes\text{I}\\ \sigma^{21}\otimes\text{I} & \sigma^{22}\otimes\text{I}\end{pmatrix}\begin{pmatrix}\text{A}\,X_1 & 0\\ 0 & \text{A}\,X_2\end{pmatrix}\right)^{-1}\begin{pmatrix}X_1^{\text{T}}\text{A} & 0\\ 0 & X_2^{\text{T}}\text{A}\end{pmatrix}\begin{pmatrix}\sigma^{11}\otimes\text{I} & \sigma^{12}\otimes\text{I}\\ \sigma^{21}\otimes\text{I} & \sigma^{22}\otimes\text{I}\end{pmatrix}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix} =$$

$$\left(\begin{pmatrix}X_1^{\text{T}}\text{A} & 0\\ 0 & X_2^{\text{T}}\text{A}\end{pmatrix}\begin{pmatrix}\sigma^{11} & \sigma^{12}\\ \sigma^{21} & \sigma^{22}\end{pmatrix}\begin{pmatrix}\text{A}\,\text{X}_1 & 0\\ 0 & \text{A}\,\text{X}_2\end{pmatrix}\right)^{-1}\begin{pmatrix}X_1^{\text{T}}\text{A} & 0\\ 0 & X_2^{\text{T}}\text{A}\end{pmatrix}\begin{pmatrix}\sigma^{11} & \sigma^{12}\\ \sigma^{21} & \sigma^{22}\end{pmatrix}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix} =$$

$$\begin{pmatrix}\sigma^{11}\,X_1^{\text{T}}\text{A}\,X_1 & \sigma^{12}\,X_1^{\text{T}}\text{A}\,X_2\\ \sigma^{21}\,X_2^{\text{T}}\text{A}\,X_1 & \sigma^{22}\,X_2^{\text{T}}\text{A}\,X_2\end{pmatrix}^{-1}\begin{pmatrix}\sigma^{11}\,X_1^{\text{T}}\text{A}\,\text{Y}_1 & \sigma^{12}\,X_1^{T}A\,\text{Y}_2\\ \sigma^{21}\,X_2^{\text{T}}\text{A}\,\text{Y}_1 & \sigma^{22}\,X_2^{T}A\,\text{Y}_2\end{pmatrix}$$

$$\text{var}\begin{pmatrix}\hat{\beta}_{1,\text{GLS}}\\ \hat{\beta}_{2,\text{GLS}}\end{pmatrix} = \begin{pmatrix}\sigma^{11}\,X_1^{T}A\,X_1 & \sigma^{12}\,X_1^{T}A\,X_2\\ \sigma^{21}\,X_2^{T}A\,X_1 & \sigma^{22}\,X_2^{T}A\,X_2\end{pmatrix}^{-1} = \begin{pmatrix}\sigma^{11}\,m_{x_1 x_1} & \sigma^{12}\,m_{x_1 x_2}\\ \sigma^{21}\,m_{x_2 x_1} & \sigma^{22}\,m_{x_2 x_2}\end{pmatrix}^{-1} =$$

$$\left(\sigma_{11}\,\sigma_{22} - \sigma_{21}^2\right)\begin{pmatrix}\sigma_{22}\,m_{x_1 x_1} & -\sigma_{12}\,m_{x_1 x_2}\\ -\sigma_{21}\,m_{x_2 x_1} & \sigma_{11}\,m_{x_2 x_2}\end{pmatrix}^{-1} = \frac{\sigma_{11}\,\sigma_{22}-\sigma_{21}^2}{\sigma_{11}\,\sigma_{22}\,m_{x_1 x_1}\,m_{x_2 x_2}-\sigma_{21}^2\,m_{x_1 x_2}^2}\begin{pmatrix}\sigma_{11}\,m_{x_2 x_2} & \sigma_{12}\,m_{x_1 x_2}\\ \sigma_{21}\,m_{x_2 x_1} & \sigma_{22}\,m_{x_1 x_1}\end{pmatrix}$$

$\text{var}(\hat{\beta}_{1,\text{GLS}}) = \frac{\left(\sigma_{11}\,\sigma_{22}-\sigma_{21}^2\right)\sigma_{11}\,m_{x_2 x_2}}{\sigma_{11}\,\sigma_{22}\,m_{x_1 x_1}\,m_{x_2 x_2}-\sigma_{21}^2\,m_{x_1 x_2}^2}$ and similarly $\text{var}(\hat{\beta}_{2,\text{GLS}}) = \frac{\left(\sigma_{11}\,\sigma_{22}-\sigma_{12}^2\right)\sigma_{22}\,m_{x_1 x_1}}{\left(\sigma_{11}\,\sigma_{22}\,m_{x_1 x_1}\,m_{x_2 x_2}-\sigma_{12}^2\,m_{x_1 x_2}^2\right)}$

(c) Using $\rho = \sigma_{12} \big/ (\sigma_{11}\, \sigma_{22})^{1/2}$ and $r = m_{x_1 x_2} \big/ (m_{x_1 x_1}\, m_{x_2 x_2})^{1/2}$ and the results in part (a) and (b), show that $\operatorname{var}(\hat{\beta}_{1,\mathrm{GLS}}) \big/ \operatorname{var}(\hat{\beta}_{1,\mathrm{OLS}}) = \left(1 - \rho^2\right) \big/ \left(1 - \rho^2\, r^2\right)$.

Solution:

$\operatorname{var}(\hat{\beta}_{1,\mathrm{GLS}}) \big/ \operatorname{var}(\hat{\beta}_{1,\mathrm{OLS}}) =$

$$\frac{\left(\sigma_{11}\, \sigma_{22} - \sigma_{21}^2\right) \sigma_{11}\, m_{x_2 x_2}}{\sigma_{11}\, \sigma_{22}\, m_{x_1 x_1}\, m_{x_2 x_2} - \sigma_{21}^2\, m_{x_1 x_2}^2} \bigg/ \frac{\sigma_{11}}{m_{x_1 x_1}} = \frac{\left(\sigma_{11}\, \sigma_{22} - \sigma_{21}^2\right) m_{x_1 x_1}\, m_{x_2 x_2}}{\sigma_{11}\, \sigma_{22}\, m_{x_1 x_1}\, m_{x_2 x_2} - \sigma_{21}^2\, m_{x_1 x_2}^2} = \frac{\sigma_{12}^2 \left(\frac{1}{\rho^2} - 1\right) \frac{m_{x_1 x_2}^2}{r^2}}{\sigma_{12}^2\, \frac{1}{\rho^2}\, \frac{m_{x_1 x_2}^2}{r^2} - \sigma_{12}^2\, m_{x_1 x_2}^2} = \frac{\frac{1}{\rho^2 r^2} - \frac{1}{r^2}}{\frac{1}{\rho^2 r^2} - 1} = \frac{1 - \rho^2}{1 - \rho^2\, r^2}$$

(d) Differentiate $\text{var}(\hat{\beta}_{12,\text{GLS}}) / \text{var}(\hat{\beta}_{12,\text{OLS}}) = (1-\rho^2)/(1-\rho^2 r^2)$ with respect to $\theta = \rho^2$ and show that this expression is a non-increasing function of θ. Similarly, differenetiate the expression with respect to $\lambda = r^2$ and show that it is a non-decreasing function of λ. Finally, compute this efficiency measure for various values of ρ^2 and r^2 between 0 and 1 at 0,1 intervals.

Solution:

$\partial\left(\frac{1-\rho^2}{1-\rho^2 r^2}\right) \Big/ \partial \rho^2 = \frac{-(1-\rho^2 r^2)+(1-\rho^2)(r^2)}{(1-\rho^2 r^2)^2} = \frac{r^2-1}{(1-\rho^2 r^2)^2}$. Both ρ^2 and r^2 are from interval $<0, 1>$ and thus the expression is non-positive for all values ρ^2 and r^2. Moreover the sign at ρ^2 is negative and this with the power of two causes that the expression is a non-increasing function in ρ^2.

$\partial\left(\frac{1-\rho^2}{1-\rho^2 r^2}\right) \Big/ \partial r^2 = \frac{\rho^2(1-\rho^2)}{(1-\rho^2 r^2)^2}$. As both ρ^2 and r^2 are from interval $<0, 1>$ this function will be non-negative for all values of ρ^2 and r^2 and thus the bigger r^2 the bigger value of the function.

Problem #8 from Baltagi, Chapter 10

a) Derive the GLS estimator for SUR with unequal number of observations given by:
$$\hat{\beta}_{\text{GLS}} = \begin{pmatrix} \sigma^{11} X_1' X_1 & \sigma^{12} X_1' X_2^* \\ \sigma^{12} X_2^{*\prime} X_1 & \sigma^{22} X_2^{*\prime} X_2^* + (X_2^{0\prime} X_2^0 / \sigma^{22}) \end{pmatrix}^{-1} \begin{pmatrix} \sigma^{11} X_1' y_1 + \sigma^{12} X_1' y_2^* \\ \sigma^{12} X_2^{*\prime} y_1 + \sigma^{22} X_2^{*\prime} y_2^* + (X_2^{0\prime} y_2^0 / \sigma^{22}) \end{pmatrix},$$
where * denotes *T* observations common for both datasets and *0* denotes extra *N* observations for the second dataset.

Solution:

First we need to define the omega matrix. From Baltagi this is block diagonal:

$\Omega^{-1} = \begin{pmatrix} \sigma^{11} I_T & \sigma^{12} I_T & 0 \\ \sigma^{12} I_T & \sigma^{22} I_T & 0 \\ 0 & 0 & \frac{1}{\sigma^{22}} I_N \end{pmatrix}$. Moreover we have: $X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2^* \\ 0 & X_2^0 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2^* \\ y_2^0 \end{pmatrix}$.

We can write: $\begin{pmatrix} X_1' & 0 & 0 \\ 0 & X_2^{*\prime} & X_2^{0\prime} \end{pmatrix} \begin{pmatrix} \sigma^{11} I_T & \sigma^{12} I_T & 0 \\ \sigma^{12} I_T & \sigma^{22} I_T & 0 \\ 0 & 0 & \frac{1}{\sigma_{22}} I_N \end{pmatrix} = \begin{pmatrix} \sigma^{11} X_1' & \sigma^{12} X_1' & 0 \\ \sigma^{12} X_2^{*\prime} & \sigma^{22} X_2^{*\prime} & \frac{1}{\sigma_{22}} X_2^{0\prime} \end{pmatrix}$.

Therefore $\hat{\beta}_{\text{GLS}} = (X' \Omega^{-1} X)^{-1} X' \Omega^{-1} y =$.
$$\begin{pmatrix} \sigma^{11} X_1' X_1 & \sigma^{12} X_1' X_2^* \\ \sigma^{12} X_2^{*\prime} X_1 & \sigma^{22} X_2^{*\prime} X_2^* + (X_2^{0\prime} X_2^0 / \sigma^{22}) \end{pmatrix}^{-1} \begin{pmatrix} \sigma^{11} X_1' y_1 + \sigma^{12} X_1' y_2^* \\ \sigma^{12} X_2^{*\prime} y_1 + \sigma^{22} X_2^{*\prime} y_2^* + (X_2^{0\prime} y_2^0 / \sigma^{22}) \end{pmatrix}$$

(b) Show that if $\sigma_{12} = 0$, SUR with unequal number of observations reduces to OLS on each equation separately.

Solution:

If $\sigma_{12} = 0$ we have: $\Omega^{-1} = \begin{pmatrix} \sigma_{11} I_T & 0 & 0 \\ 0 & \sigma_{22} I_T & 0 \\ 0 & 0 & \sigma_{22} I_N \end{pmatrix}$ and thus $\Omega^{-1} = \begin{pmatrix} \frac{1}{\sigma_{11}} I_T & 0 & 0 \\ 0 & \frac{1}{\sigma_{22}} I_T & 0 \\ 0 & 0 & \frac{1}{\sigma_{22}} I_N \end{pmatrix}$ so that $\sigma^{12} = 0$ and $\sigma^{\text{ii}} = \frac{1}{\sigma_{\text{ii}}}$ for i = 1,2.

From previous example we have $\hat{\beta}_{\text{GLS}} = (X' \Omega^{-1} X)^{-1} X' \Omega^{-1} y = \begin{pmatrix} \frac{X_1' X_1}{\sigma_{11}} & 0 \\ 0 & \left(\frac{X_2^{*\prime} X_2^*}{\sigma_{22}} + \frac{X_2^{0\prime} X_2^0}{\sigma_{22}}\right) \end{pmatrix}^{-1} \begin{pmatrix} \frac{X_1' y_1}{\sigma_{11}} \\ \frac{X_2^{*\prime} y_2^* + X_2^{0\prime} y_2^0}{\sigma_{22}} \end{pmatrix}$

and $\hat{\beta}_{\text{GLS}} = (X' \Omega^{-1} X)^{-1} X' \Omega^{-1} y = \begin{pmatrix} (X_1' X_1)^{-1} X_1' y_1 \\ (X_2^{*\prime} X_2^* + X_2^{0\prime} X_2^0)^{-1} (X_2^{*\prime} y_2^* + X_2^{0\prime} y_2^0) \end{pmatrix} = \begin{pmatrix} \hat{\beta}_{1,\text{OLS}} \\ \hat{\beta}_{2,\text{OLS}} \end{pmatrix}$