Charles University in Prague
Faculty of Social Sciences

BACHELOR THESIS

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Oligopolistic Markets in Terms of Equilibrium Problems with Equilibrium Constraints

Institute of Economic Studies

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Study program: Economic Theories

Prohlašuji, že jsem svou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 31. 5. 2006 Mgr. Michal Červinka

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ABSTRACT
Title: Oligopolistic Markets in Terms of Equilibrium Problems with Equilibrium Constraints
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Abstract: In 2001 the brand-new concept of solving a situation on the deregulated electricity markets under an independent system operator (ISO) regime was presented. On this markets there are several energy producers supplying the market in the position of the market leaders and there is also the ISO possing as the single market follower. This game can be mathematically represented as a problem from the newly-emerged class of mathematical problems called Equilibrium Problems with Equilibrium Constraits (EPECs).

This bachelor thesis should serve as a state-of-the-art overview, i.e., the goal of this thesis is to map the existing literature on EPECs. The mathematical formulation of the deregulated electricity market problem under ISO regime is presented along with the general formulation of EPECs. We show the problematic intrinsic features of EPECs (e.g. lack of convexity, huge computational complexity) in terms of simple examples. In the end of the thesis the existing computational approaches are discussed.

ABSTRAKT
Název práce: Oligopolistic Markets in Terms of Equilibrium Problems with Equilibrium Constraints
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Abstrakt: V roce 2001 byl uveden zcela nový pohled na řešení situací na deregulovaných trzích s elektřinou s režimem ISO (independent system operator). Na těchto trzích působí výrobci energie, kteří zásobují trh, jako tržní vůdci a ISO vystupuje jako hráč v pozici následovníka. Tuto hru lze matematicky zapsat jako úlohu z nové třídy matematických problémů, tzv. ekvilibriálních úloh s ekvilibriálními omezeními (EPEC).

Z této bakalářské práce by měl čtenář poznat přehled současného vývoje, tzn. cílem práce je zpracovat současnou dostupnou literaturu o EPECích. Čtenář zde najde matematickou formulaci jak úlohy deregulovaného trhu s elektřinou s režimem ISO, tak i obecného EPECu. Na příkladech pak postupně probíráme hlavní problematické charakteristiky těchto úloh (chybějící konvexitu, náročnou výpočetní složitost apod.). V závěru práce popisujeme existující postupy řešení.

Contents

1 Introduction

Namely, because the shape of the whole universe is most perfect and, in fact, designed by the wisest creator, nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth.

Leonhard Euler (1744)

The central theme of considerable part of everything that has been written down is theme of a conflicting situation, a collision of interests. Though being one of the oldest notions among mankind, the scientific approach has started relatively recently, around nineteen hundred and thirty. Since then, more and more scientific disciplines, like applied mathematics, economics, sociology and politics, devote attention to the analysis of conflicting situations.

An individual has to make a decision and each possible decision leads to different outcome. More, this individual may not be the only one who decides about the particular outcome. The application of "game theory" mainly deals with economic (that is of the main interest in this work) and political conflicting situations, worst case designs and modelling of war games.

In multi-person decision making, optimality is not a well-defined concept. Therefore, we can consider several different notions. However, there are two distinct widely used equilibrium concepts: a solution to a noncooperative game, where each player can't improve his or her outcome by altering his or her decision unilaterally, is called a *Nash equilibrium solution*, and in case when cooperation is allowed, if no other other joint decision of the player can improve the performance of at least on of them, without degrading the performance of the other, we arrive at a *Pareto optimal solution*.

Dominant firms in a market can exercise their power to manipulate the market to their own advantage. If all firms have the same market share then the market can be modelled by a Cournot-Nash equilibrium concept [11], a noncooperative game in which all firms simultaneously compete against each other. When there is a single dominant firm, however, the market must be modelled as a Stackelberg (or *single-leader-follower*) game [16], in which the dominant firm (usually due to some temporal advantage over the other firms), the market *leader*, maximizes its profit subject to all other firms, the *followers*, being in a competitive equilibrium. Mathematically, Stackelberg games are usually expressed in terms of bilevel optimization problems called *mathematical problems with equilibrium constraints* (MPECs). This class of problems introduced in 1970s was under heavy interest of several mathematicians in 1990s and we can already find several monographs on MPECs (e.g. [7] and [12]).

Between these two sort of extreme situations lays the *multi-leader-follower* game that has multiple dominant firms and a number of followers. This situation might occur, e.g., if the standard Cournot-Nash strategy is simultaneously deserted by two or more firms. In such a case each of them has to make some assumptions not only about the behavior of the followers, but also of the remaining leaders. Concerning the behavior of the leaders, one can distinguish two situations: first that all leaders cooperate and second that the leaders' strategies form the Cournot-Nash equilibrium on the upper level. The former case has been investigated in [8] and [10] using tools of the multiobjective optimization and the generalized differential calculus by Boris Mordukhovich. In latter case, one obtains a rather complex equilibrium problem of parametrically coupled MPECs. In this thesis we shall focus on both cases. As to our knowledge, there has not yet been published a work that investigates situations where the leaders build some coalitions.

Within the class of games, in which leaders act noncooperatively, we shall consider only those in which the followers' responses are constrained to be identical for each leader, so called *multi-leader-identical-follower* games. Problems of this type arise, for example, in the analysis of deregulated electricity markets [5]. To mathematically express these problems, one can use the novel modelling paradigm of *equilibrium problems with equilibrium constraints* (EPECs). This class of equilibrium problems was introduced in [14] and since then it was further developed, e.g., in [13], [3], [6] and [17]. In some papers (e.g. [6], [17]) there are already proposals how to treat EPECs in order to solve them. Several of the solving techniques can be found there.

This bachelor thesis is organized as follows. In the next section we will introduce the above mentioned optimization and generalized differential tools along with other mathematical objects that we will use throughout the thesis. In section 3 we will describe the model of competition in deregulated electricity markets as it was published in [5]. This model in fact motivated the researchers to investigate EPECs more thoroughly. In section 4 we will give the rigorous definition of a noncooperative solution to EPECs and discuss some intrinsic (usually problematic) features of this class of problems and we will illustrate EPECs via some simple academic examples. We will also discuss there some of already existing solution methods. Section 5 is devoted to cooperative equilibrium bilevel games. We present the numerical approach developed by the author of this thesis along with results of the application to academic examples of oligopolistic markets.

2 Preliminaries

In this section we introduce the notation and give definitions to mathematical objects that are used in this thesis. We also present briefly their main properties to help the reader to properly understand them.

By $\mathbb{R}^n_+$ we mean the nonnegative orthant of $\mathbb{R}^n$, i.e., $y \in \mathbb{R}^n_+$ is equivalent to $y_i \geq 0, i = 1, \ldots, n$. The set operator riK denotes the *relative interior* of a set K. Recall that x is a relative interior point of C, if x is an interior point of C relative to $affC$, where $affC$ is an affine space generated by C. A mapping F is called a *set-valued map* or *multifunction* if F maps points from a space X to subsets of a space Y. Multifunctions are usually denoted as $F[X \rightrightarrows Y]$. We will benefit from notation a_{-i} that denotes a vector constructed from a vector a by omitting the ith component. E.i., $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n), i = 1, \ldots, n$.

We say that $F[\Omega \subset \mathbb{R}^n \to \mathbb{R}^m]$ is (globally) Lipschitz continuous with modulus $k > 0$ on Ω, if

$$\| F(x_1) - F(x_2) \| \leq k \| x_1 - x_2 \| \quad \text{for all } x_1, x_2 \in \Omega.$$

We say that F is locally Lipschitz on Ω if it is Lipschitz on an $\epsilon > 0$ neighbourhood of each point $x \in \Omega$. The notion of Lipschitz continuity plays a fundamental role in variational analysis; subdifferential theory even characterizes the presence of Lipschitz continuity and provides a calculus. Moreover, every convex function is locally Lipschitz.

Clarke's directional derivative for a locally Lipschitz function $f[\mathbb{R}^n \to \mathbb{R}]$ with Lipschitz constant k is defined by

$$f^\circ(\bar{x}; v) = \limsup_{x \to \bar{x}, t \searrow 0} \frac{f(x + tv) - f(\bar{x})}{t}$$

and Clarke's generalized gradient or *Clarke subdifferential* of f at $\bar{x}$

$$\bar{\partial} f(\bar{x}) = \{x^* \mid \langle x^*, v \rangle \leq f^\circ(\bar{x}; v) \;\; \forall v \in \mathbb{R}^n\}.$$

For arbitrary x, $f^\circ(x, \cdot)$ is convex, positively homogeneous and globally Lipschitz with constant k. Further

$$\bar{\partial} f(\cdot) \text{ is convex and nonempty}$$

and $\bar{\partial} f(\cdot) \subset k\mathbb{B}$, where $\mathbb{B}$ is a closed unit ball in $\mathbb{R}^n$ with radius 1,

$$\bar{\partial} \left(\sum_i (\alpha_i f_i)(\bar{x}) \right) \subset \sum_i \alpha_i \bar{\partial} f_i(\bar{x}) \tag{1}$$

$$f^\circ(x; v) = (-f)^\circ(x; -v), \qquad \bar{\partial}(-f)(\bar{x}) = -\bar{\partial} f(\bar{x})$$

$$f^\circ(\bar{x}; v) = \max_{x^* \in \bar{\partial} f(\bar{x})} \langle x^*, v \rangle,$$

$$\text{if } \hat{x} \text{ is a local minimizer of } f \text{ then } \;\; 0 \in \bar{\partial} f(\hat{x}). \tag{2}$$

A very important result is that Clarke subdifferential is the smallest one among any convex-valued subdifferentials with properties (1), (2) and "robustness" (closed graph)

$$\bar{\partial} f(\bar{x}) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \bar{\partial} f(x),$$

the symbol “Lim sup” stands for the Painlevé-Kuratowski upper (or outer) limit that is defined for a set-valued mapping $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ at a point $\bar{x}$ by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \{y \in \mathbb{R}^m | \exists x_k \to x, \exists y_k \to y \text{ with } y_k \in F(x_k)\}.$$

Recall that $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. Given $\Omega \subset \mathbb{R}^n$ and $\bar{x} \in \text{cl}\Omega$, the *(basic, limiting) normal cone* to Ω at $\bar{x}$ is defined by

$$N_\Omega(\bar{x}) = \operatorname*{Lim\,sup}_{x \to \bar{x}} [\text{cone } (x - \Pi_\Omega(x))], \tag{3}$$

where “cone” stands for the conic hull of a set, and where $\Pi_\Omega(\cdot)$ denotes the Euclidean projector onto $\text{cl}\Omega$, i.e.,

$$\Pi_\Omega(x) := \{w \in \text{cl}\Omega \,|\, \| x - w \| = \text{dist}(x, \Omega)\}.$$

It is well known that our basic normal cone (3) is generally *nonconvex*, while for a convex set Ω it reduces to the normal cone in the sense of convex analysis. The normal cone can be equivalently represented as

$$N_\Omega(\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \hat{N}_\Omega(x),$$

where the prenormal, or Fréchet normal, cone $\hat{N}_\Omega(\cdot)$ is defined by

$$\hat{N}_\Omega(\bar{x}) := \begin{cases} \{x^* \in \mathbb{R}^n | \operatorname*{limsup}_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \le 0\} & \text{for } \bar{x} \in \text{cl}\Omega, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the prenormal cone is the negative polar cone to the Bouligand-Severi contingent cone, while the normal cone (3) cannot be dual to any tangent cone due to its nonconvexity; polar cones are always convex.

As an example consider $N_{\mathbb{R}_+}(\cdot)$. At $x = 0$ it is $\mathbb{R}_-$, at $x > 0$ it is just 0 and at $x < 0$ it is empty set. If we, for example need to find local minimizers of a continuously differentiable function f on $\mathbb{R}_+$, these points need to satisfy

$$0 \in f'(x) + N_{\mathbb{R}_+}(x),$$

which exactly corresponds to the know rule that at the interior points, the derivative has to vanish, and at 0 (the “left” boundary point) the function needs to be nondecreasing.

In nonlinear optimization problems (NLP) of the form

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } h_i(x) = 0, i = 1, \ldots, p, \\ &\qquad\qquad g_k(x) \leq 0, k = 1, \ldots, m, \end{aligned} \tag{4}$$

where functions $f, h_i, i = 1, \ldots, p$, and $g_k, k = 1, \ldots, m$, are continuously differentiable at some reference point $\bar{x}$, the necessary optimality conditions for $\bar{x} \in \mathbb{R}^n$ to be a solution of (4) can be expressed in the form of so called *Karush-Kuhn-Tucker system of conditions* (KKT)

$$\begin{aligned} &h_i(\bar{x}) = 0, i = 1, \ldots, p, \;\; g_k(\bar{x}) \leq 0, k = 1, \ldots, m, \\ &\text{there exists vectors } \bar{v} \in \mathbb{R}^p, \bar{u} \in \mathbb{R}^m \text{ such that} \\ &\bar{u}_k g_k(\bar{x}) = 0, k = 1, \ldots, m, \;\; \bar{u}_k \geq 0, k = 1, \ldots, m, \\ &\nabla_x L(\bar{x}, \bar{u}, \bar{v}) = \nabla f(x) + \sum_{i=1}^{p} \bar{v}_i \nabla h(\bar{x}) + \sum_{k=1}^{m} \bar{u}_k \nabla g_k(\bar{x}) = 0. \end{aligned}$$

A function L is called Lagrangian function and u, v are called *Lagrangian multiplicators* or simply multiplicators. In the case when (4) is a stricly convex NLP, i.e., the corresponding Lagrangian function is strictly convex in variable x, the KKT conditions at $\bar{x}$ are necessary and sufficient conditions for $\bar{x}$ to be a local solution to our NLP. If we however denote the feasible set by

$$Z = \{x \in \mathbb{R}^n | h_i(x) = 0, i = 1, \ldots, p, g_k(x) \leq 0, k = 1, \ldots, m\}$$

the necessary optimality conditions can be equivalently expressed via *generalized equation* (GE)

$$0 \in \nabla f(\bar{x}) + N_Z(\bar{x}).$$

The symbolical notation is more elegant and (perhaps unfortunately) we no longer need to define the multiplicators (however, the notion is captured by the structure of the limiting normal cone, see the simple example above).

Finally we need to define a *mixed strategy*. This concept comes from the game theory. It is a strategy of a player consisting of feasible moves and a probability distribution (in case of a discreet problem just a collection of weights) which corresponds in case of a repeatedly played game to how frequently each move is to be played. On the other hand, *pure strategy* is a special case of mixed strategy that is a Dirac measure, a probability measure which in terms of probability represents the almost sure outcome x in the sample space X. A player can use a mixed strategy, e.g., when he or she is indifferent between several pure strategies, or when keeping the opponent guessing is desirable. The concept of mixed strategies is also used in situations when the existence of a solution (pure strategy) to a game is not ensured and we could use a generalized concept of solution.

3 Competition in Deregulated Electricity Markets

An important issue in all deregulated electricity markets is the market power of participants such as generators, large utilities, or providers of ancillary services. The transportation of power from a generation node (source) to a consumption node (sink) is governed by the Kirkhoff Laws (laically speaking, power flows along the paths of the least resistance). So, transmission of power is different from the transportation of the ordinary commodity in a spatial market. The location and quantity of any injection or withdrawal of power determines the actual transmission capacity of any link in electric network. As a result, the key issue in the overall design is how a network (grid) operator dispatches electricity. The difference is particularly marked when the network contains loops and there are transmission capacity limits.

In this chapter that is based on [5] we show how to model so called *pool-type* markets as operated in Australia, New Zealand and some parts of United States, where the *independent system operator* (ISO) performs the following functions: 1) dispatching electricity from generators to consumers by maximizing "social welfare" based on the cost/benefit functions that are bid by generators and consumers given the ISO is constrained by transmission limits, security considerations and other operating constraints; 2) facilitating financial settlements based on locational marginal prices.

3.1 Problem Formulation: ISO's pricing and dispatch problem

Suppose an electric network with $N+1$ nodes (buses) labelled $0, \ldots, N$, and a set L of links, where the link between node i and j is written ij.

Bidders (generators and retailers) have complete information about the network, the ISO's operation procedure and all other participants' cost/utility functions. The bids are submitted to ISO in form of their supply or demand functions. The ISO (taking account of the network) solves a social welfare maximization problem assuming the bids are truthful, announcing a dispatch for each bidder and possibly distinct prices at each node. Consumers pay generators according to the scheduled dispatch and nodal prices. The market is then cleared according to each player's binding bid.

For the sake of simplicity, assume that there is a single generator or consumer at any Node i. Otherwise we may state that generators or consumers located in the same node are assumed to have identical cost or utility functions (but as active bidders each bids independently).

Each player's cost or utility function is a quadratic function in quantity q_i, either cost, $A_i q_i + B_i q_i^2 \, (q_i \geq 0)$, or utility, $-A_i q_i - B_i q_i^2 \, (q_i \leq 0)$, where each A_i and B_i is assumed to be positive. We can hence let generators (consumers) bid their supply (demand) functions to the ISO in the form of a pair of coefficients (a_i, b_i). To avoid any confusion, recall that $A_i q_i + B_i q_i^2$ and $a_i q_i + b_i q_i^2 \, (q_i \geq 0)$, are cost and supply functions of the generator, respectively, and $-A_i q_i - B_i q_i^2, -a_i q_i - b_i q_i^2 \, (q_i \leq 0)$, are the utility and demand functions of the consumer, respectively. A consumer at Node i is dispatched a quantity in the range $[0, \frac{A_i}{2B_i}]$, where his or her utility is nondecreasing in the quantity consumed.

The ISO solves the following problem of minimizing the social cost (same problem as maximizing the social welfare) over all $N+1$ nodes:

$$\begin{aligned}
&\underset{q_0,\ldots,q_N}{\text{minimize}} \sum_{i=0}^{N}(a_i q_i + b_i q_i^2) \\
&\text{subject to } q_0 + q_1 + \cdots + q_N = 0 && (5)\\
&\qquad -C_{ij} \leq \sum_{k=1}^{N} \phi_{ij,k} q_k \leq C_{ij},\ i<j, ij \in L && (6)\\
&\qquad q_i \geq 0,\ i:\ \text{generator},\ q_i \leq 0,\ i:\ \text{consumer}, && (7)
\end{aligned}$$

where C_{ij} denotes the transmission limit on the link ij and $\phi_{ij,k}$ (distribution factor) denotes the contribution of an injection (or withdrawal) at Node k to the link ij. The distribution factors are determined by the network's physical properties. The optimal solution to this problem is denoted by $q = (q_0(a,b), \ldots, q_N(a,b))$.

Let us denote the Lagrange multipliers corresponding to the optimal solution such that $\lambda, \underline{\mu}_{ij}, \overline{\mu}_{ij}$ and ν_i are the multiplier corresponding to (5), (6) (left hand side and right hand side inequality) and (7), respectively. The optimal quantities q_k commit the ISO to paying (charging) the kth player a price that is consistent with their bid supply (demand) function:

$$p_k = a_k + 2b_k q_k.$$

Additionally, if $q_k \neq 0$, the ISO in effect sets p_k equal to

$$p_k = -\lambda - \sum_{i<j, ij\in L} (\overline{\mu}_{ij} - \underline{\mu}_{ij})\phi_{ij,k},$$

as it comes from the ISO's Karush-Kuhn-Tucker (KKT) conditions. Note that when binding transmission quantities are missing, p_k equals the shadow price $(-\lambda)$ of the requirement that electricity generated equals electricity consumed.

The authors of [5] state several remarks. In the proposed model, players bid continuous supply or demand functions. In real electricity markets, the format of bids is typically a stepwise function, which is difficult to formally model. As an example, in Australia, generators bid sixteen hours in advance ten steps (price/quantity pairs) for each of 48 half hours of the next day. The ISO then predispatches the generators according to the forecast loads for each of the 48 half hours. However, up to five minutes before the real-time dispatch, generators are allowed to change their quantity, but not price, offers by moving their quantities up or down along the offered price stack. This model neither captures the stepwise bids nor the rebidding aspect of that market.

3.2 Problem Formulation: market participant's bidding problem

Let us now consider the behavior of profit-maximizing player on the market described above. The question is, what supply or demand function should Bidder i submit to the ISO to achieve the player's own maximal profit?

Given that Bidder i's price is $a_i + 2b_iq_i$, then the profit maximization problem is:

$$\begin{aligned}
\underset{a_i,b_i}{\text{maximize}}\ & (a_i + 2b_iq_i)q_i - (A_iq_i + B_iq_i^2) \\
\text{subject to}\ & \underline{A}_i \le a_i \le \overline{A}_i \\
& \underline{B}_i \le b_i \le \overline{B}_i \\
& q_i \text{ such that } q = (q_0, \ldots, q_N) \text{ solves the ISO's minimization} \\
& \text{problem given the other participants' bids } (a_{-i}, b_{-i}).
\end{aligned} \tag{8}$$

The constants $\underline{A}_i, \overline{A}_i$ and $\underline{B}_i, \overline{B}_i$, assumed to satisfy

$$0 < \underline{A}_i \le A_i \le \overline{A}_i \text{ and } 0 < \underline{B}_i \le B_i \le \overline{B}_i$$

are lower and upper bounds for a_i and b_i that are based on industry knowledge and are imposed by the ISO.

The outlined problem is a bilevel program, where the lower-level problem is that q must solve the optimal power flow problem of the ISO. One of the effective tricks to better approach these problems is to replace the constraints in (8) with the KKT conditions of the ISO's problem, that is, to reformulate the bilevel problem for the ith Bidder as an MPEC.

3.3 Problem Formulation: EPEC

Since the ISO's problem is a strictly convex quadratic problem, a dispatch $q(a,b)$ solves the problem if and only if there exist multipliers corresponding to the constraints that satisfy the usual KKT conditions at $q(a,b)$. Given the other participants' bids (a_{-i}, b_{-i}), Bidder i's problem becomes

$$\begin{aligned}
\underset{a_i,b_i,q,\lambda,\underline{\mu},\overline{\mu},\nu}{\text{maximize}}\ & (a_i + 2b_iq_i)q_i - (A_iq_i + B_iq_i^2) \\
\text{subject to}\ & \underline{A}_i \le a_i \le \overline{A}_i \\
& \underline{B}_i \le b_i \le \overline{B}_i \\
\text{(ISO-KKT)}\ & \begin{cases}
\text{where, for each } j = 0, \ldots, N \text{ and } mn \in L \text{ with } m < n: \\
a_j + 2b_jq_j + \lambda + \sum_{i<k, ik\in L} \phi_{ik,j}(\overline{\mu}_{ik} - \underline{\mu}_{ik}) - \nu_j = 0 \\
q_0 + \cdots + q_N = 0 \\
0 \le C_{mn} + \sum_{k=1}^{N} \phi_{mn,k}q_k \perp \underline{\mu}_{mn} \ge 0 \\
0 \le C_{mn} - \sum_{k=1}^{N} \phi_{mn,k}q_k \perp \overline{\mu}_{mn} \ge 0 \\
0 \le q_j \perp \nu_j \ge 0 \ \text{ if Bidder } j \text{ is a generator} \\
0 \ge q_j \perp \nu_j \le 0 \ \text{ if Bidder } j \text{ is a consumer.}
\end{cases}
\end{aligned} \tag{9}$$

Since the equilibrium conditions in (9) are only in the form of the complementarity constraints, we refer to this type of MPEC as to the MPCC. The game based on all the participants' problems (9), for $i = 0, \dots N$, gives rise to an EPEC. One can form from the KKT conditions for each bidder a system of KKT conditions in which the ISO's KKT conditions appear only once. Authors of [5] call this system All-KKT (which constitutes an EPCC) and search for a Nash stationary equilibrium as a solution of All-KKT. For mode details on deregulated electricity markets and results of the numerical tests on several examined games we refer to the original work.

In the next section, we define EPEC and particularly EPCC in the general form and instead of KKT conditions we will use generalized equations.

4 Noncooperative Equilibrium Problems with Equilibrium Constraints

One usually defines oligopoly as market with “few sellers” (which is indeed the exact meaning of the word oligopoly). This market structure comes forth when a firm believes that the outcome of its decision making depends heavily on the decisions taken by other market player. So the firm seeks to maximize its profit (or minimize its losses) in the situation of close interdependence. The theory of oligopoly investigates this process of decision making, with interest to make predictions of decisions of sellers in such situations.

The application of game theory (that puts stress on the rationality of calculations made by the firm concerned) to oligopoly theory has led to fundamental reinterpretations of the models. This was one of the major impulses for both mathematical and economical research communities. Mathematicians soon found out that these patterns of rational reasoning may be applied to other scientific areas, such as biology (when studying natural equilibria of populations of symbiotic species or predator-prey related species), physics (problems of mechanics; contact problems etc.) and other social sciences (sociology, politics and demography).

The development and continuous process of generalization of game theoretical models recently led to the formulation of a special mix of Nash game and Stackelberg game, in which Nash equilibrium structure appears on both levels. In this section we will present the standard mathematical formulation of the noncooperative solution to a general EPEC. We will also show how to choose the parameters of our general model in order to get the EPCC from the previous section.

4.1 Mathematical formulation

Assume that we have to do with n leaders and m followers. Let x_i denote the strategy of the ith leader and the vector $x := (x_1, x_2, \dots, x_n)$ contain the strategies of all leaders. We require that $x \in \omega$, where $\omega \subset \mathbb{R}^{nl_1}$ is the set of *feasible leaders' strategies.* We can write $\omega = \bigtimes_{i=1}^{n} U_i$, where $U_i \subset \mathbb{R}^{l_1}$ is the set of feasible strategies for the ith leader, but it can be defined also by joint constraints, across all the leaders' strategies.

Analogously, $y := (y_1, y_2, \dots, y_m) \in \mathbb{R}^{ml_2}$ consists from all the followers' strategies. A vector y is feasible, provided its all components $y_j, j = 1, \dots, m$, belong to the sets $V_j \subset \mathbb{R}^{l_2}$. The followers act according to their objectives $f_j[\mathbb{R}^{nl_1+ml_2} \to \mathbb{R}], j = 1, 2, \dots, m$. For a given vector $\bar{x} \in \omega$ and given strategies $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{j-1}, \bar{y}_{j+1}, \dots, \bar{y}_m$ the strategy of the jth follower amounts to a solution of the optimization problem

$$\begin{array}{ll} \text{minimize} & f_j(\bar{x}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{j-1}, y_j, \bar{y}_{j+1}, \dots, \bar{y}_m) \\ \text{subject to} & \\ & y_j \in V_j \end{array} \tag{10}$$

in variable y_j. We will assume that all objectives $f_j, j = 1, 2, \dots, m$, are twice continuously differentiable. Under this assumption it is clear that, given $\bar{x}$, a corresponding vector $\bar{y}$

amounts to a solution of the mixed complementarity problem (MCP)

$$0 \in F(\bar{x}, y) + N_\Omega(y), \tag{11}$$

where

$$F(x, y) := \begin{bmatrix} \nabla_{y_1} f_1(x, y) \\ \nabla_{y_2} f_2(x, y) \\ \vdots \\ \nabla_{y_m} f_m(x, y) \end{bmatrix} \text{ and } \Omega := \bigtimes_{i=1}^{m} V_i.$$

Note, that this mixed complementarity problem can be equivalently rewritten to the KKT conditions. The advantage of MCP over KKT is that do not have to deal with the multiplicators anymore.

Let S denote the decision rule of the followers (the solution map), i.e.,

$$S(x) := \{y \in \mathbb{R}^{ml_2} | \, 0 \in F(x, y) + N_\Omega(y)\}.$$

In microeconomics, to find a noncooperative solution to Nash game one usually derives so called *response function*. E.i., one works with an implicit assumption, that the players react uniquely. This basic assumption can be written down in terms of EPECs in the following form

(A1) For all $x \in \omega, S(x)$ is either empty or a singleton.

This can be ensured, e.g., by the requirement that all objectives f_j are strictly convex in y_j on $\mathbb{R}^{l_2}$ for all admissible values of the remaining arguments. In case (A1) holds we may call $S(x)$ the optimal response function. This assumption, of course, does not ensure that (11) possesses a solution for each admissible vector $\bar{x}$.

The behavior of the leaders is described by their individual objectives $\varphi_i[\mathbb{R}^{nl_1+ml_2} \to \mathbb{R}], i = 1, 2, \ldots, n$. We say that $\bar{x}$ is a noncooperative solution to EPEC if for all $i = 1, \ldots, n$, $\bar{x}_i$ is a solution to the optimization problem

$$\begin{array}{ll} \text{minimize} & \varphi_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n, y) \\ \text{subject to} & \\ & 0 \in F(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n, y) + N_\Omega(y), \\ & x_i \in U_i. \end{array} \tag{12}$$

This is another way how to approach the bilevel structure of the problem (in previous chapter, we modelled the optimal behavior of the follower via KKT conditions).

It is now clear that if we set parameters $n := N+1, m := 1, l_1 := 2, l_2 := N+1$, sets

$$U_i := \{x_i = (a_i, b_i)^T | \underline{A}_i \le a_i \le \overline{A}_i, \underline{B}_i \le b_i \le \overline{B}_i\}, i = 0, \ldots, N,$$

$$\begin{aligned} \Omega = V_1 := \{y = (q_0, \ldots, q_N)^T | q_0 + q_1 + \cdots + q_N = 0, \\ - C_{ij} \le \sum_{k=1}^{N} \phi_{ij,k} q_k \le C_{ij}, \ i < j, ij \in L, \\ q_i \ge 0, \ i: \text{ generator}, \ q_i \le 0, \ i: \text{ consumer}\}, \end{aligned}$$

and functions

$$\varphi_i(x_0, \ldots, x_N, y) := (A_i y_{i+1} + B_i y_{i+1}^2) - (x_{i1} + 2x_{i2} y_{i+1}) y_{i+1}, i = 0, \ldots, N,$$
$$f(x_0, \ldots, x_N, y) := \sum_{i=0}^{N} x_{i1} y_{i+1} + x_{i2} y_{i+1}^2$$

then we arrive precisely at the EPEC corresponding to the deregulated electricity market dispatch problem.

We strengthen the assumption (A1) to the following form

(A1') S is single-valued and locally Lipschitzian on an open set containing ω.

Under this assumption we can reduce the EPEC to the game among leaders only. That means, $\bar{x}_i$ is a solution to optimization problem

$$\begin{array}{ll} \text{minimize} & \theta_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) \\ \text{subject to} & \\ & x_i \in U_i, \end{array} \tag{13}$$

where functions $\theta_i[\omega \to \mathbb{R}]$ are defined by

$$\theta_i(x) = \varphi_i(x, S(x))$$

and we will call them cost functions of the reduced game among leaders only. In the case of non-hierarchical game, θ_i would denote the objective function of the ith player.

This implicit functional representation is advantageous for the application of the classical results from the Nash game theory. However, one needs to be always aware of the implicitly incorporated lower level structure that can make the problem much more difficult and not so straightforward to solve.

4.2 Intrinsic features

In this subsection we will illustrate several features of EPECs that make problems of this class of optimization problems challenging. One of the setbacks that have to be taken into account is the nonuniqueness of the lower-level problem. Recall, that in our formulation we overcame the problem by imposing an additional assumption. For illustration, we construct an EPEC with the set-valued lower-level solution map in Example 1.

Example 1

Consider the three-person game on $[0, 1]^3$ with player 1 and 2 in the role of leaders given by the cost functions

$$\begin{aligned} \varphi_1(x_1, x_2, y) &= (x_1)^2 y + x_2, \\ \varphi_2(x_1, x_2, y) &= (x_2)^2 y + x_1, \\ f(x_1, x_2, y) &= (x_1 + x_2 - 1) y^2. \end{aligned}$$

One can easily check that the lower-level solution mapping is in the form

$$S(x_1, x_2) = \begin{cases} \{1\} & \text{when } x_1 + x_2 < 1; \\ [0,1] & \text{when } x_1 + x_2 = 1; \\ \{0\} & \text{when } x_1 + x_2 > 1; \end{cases} \tag{14}$$

Hence without any additional assumption on the behavior of follower, it is not clear, which strategy should leaders expect to be played by follower if $x_1 + x_2 = 1$.

From the theory on Nash equilibria we know that the equilibrium point in non-hierarchical game exists if feasible sets of all players are convex and compact and all the objectives are continuous and convex. The following example will show how easily the solution of EPEC may not exist even in the case when assumptions of convexity appear to be satisfied.

Example 2

Let us consider the following three-person game on $[0,2]^2 \times [-2,2]$ with cost functions

$$\begin{aligned} \varphi_1(x_1, x_2, x_3) &= (x_1 - x_2)^2 - x_1 + x_2 + x_3 + k_1, \\ \varphi_2(x_1, x_2, x_3) &= (x_1 - x_2)^2 - 2(x_3)^2 + k_2, \\ \varphi_3(x_1, x_2, x_3) &= (x_3)^2 - 2x_1x_3 + 2x_2x_3 + k_3, \end{aligned}$$

where $k_i, i = 1, \ldots, 3$, are arbitrary real constants.

Obviously, as a three person Nash game, this setting satisfies assumptions for existence of an equilibrium solution. Consider now that first two players become the leaders. The game can be reduced to game among the leaders only with cost functions

$$\begin{aligned} \theta_1(x_1, x_2) &= (x_1 - x_2)^2 + k_1, \\ \theta_2(x_1, x_2) &= -(x_1 - x_2)^2 + k_2, \end{aligned}$$

and with the optimal response function of the follower $S(x_1, x_2) = x_1 - x_2, (x_1, x_2) \in [0,2]^2$. This game with derived solution can be found in [2]. Even though we are now unable to find a solution (the response functions do not intersect) since the cost function of the second player is strictly concave, we can still find a solution in mixed strategies. To ensure the existence of mixed strategy solution only the compactness of feasible sets and continuity of objectives are required. The solution to our EPEC can now be derived as

$$\begin{aligned} x_1^* &= 1 && \text{with probability } 1, \\ x_2^* &= \begin{cases} 0 \\ 2 \end{cases} && \begin{matrix} \text{with probability } 1/2, \\ \text{with probability } 1/2, \end{matrix} \\ x_3^* &= x_1^* - x_2^* && \text{with probability } 1. \end{aligned}$$

We purposely write the solution of the third player in that form to emphasize that he always plays pure strategy even though it is 1 with probability 1/2 and −1 with probability

1/2. This example also nicely illustrates the fact that even under convex setting, EPECs generally lead to nonconvex optimization problems.

From the mathematical formulation of EPEC in subsection 4.1 one can as well see that we can construct EPEC via parameterized MPECs. Each leader's problem is indeed MPEC with parameters that are the solutions of other MPECs of similar form representing other leaders' decision-making. Solving of an MPEC and even parametrized MPEC is nowadays a well described and understood procedure. However, EPEC forms a huge and complex structure of MPECs tight up together with parameters which makes it in general extremely computationally demanding to solve.

4.3 Numerical approaches

As mentioned at the end of section 3, the authors of [5] try to solve the All-KKT reformulation of EPEC. In the second part of their paper one can find the examples and numerical results.

The All-KKT system can be generally difficult to solve. The sources for this may be the forms of objective functions of the leaders and the followers and even the number of players on both levels. Recall, that the more leaders the more KKT systems is being coupled into one All-KKT.

The most natural and convenient method is so called *diagonalization method* which leverages existing MPEC solvers and is very easy to implement. It is an iteration method, starting at some initial point and looping over each MPEC. Each time one MPEC (one leader's problem) is solved via MPEC algorithms in one variable for fixed decision variables of the remaining leaders. Then the vector of leaders' strategies is "updated" and another MPEC is being solved. If the improvement is below some accuracy level, we get our solution.

No matter how much the cyclic procedure of improving a vector of solutions on the upper level by solving one MPEC at the time sounds appealing, however, there is no guarantee that the procedure converges. We would need some form of fixed point theorem. To our knowledge, every version of fixed point theorem needs convexity that unfortunately is generally not present in EPECs, see Example 2.

Since the most problematic part of KKT system in MPECs are the complementarity conditions, in [17] the reader can find another method, *sequential nonlinear complementarity algorithm*, based on relaxation of these constraints simultaneously in each MPEC. This idea is based on a method developed for solving MPECs via solving a sequence of nonlinear programs. To solve EPEC by the method proposed by Su, one needs to solve a sequence of nonlinear complementarity problems (NCPs), hence the name of the method. In [17] the reader can also find a comparison of the diagonalization method with sequential NCP algorithm on randomly generated EPEC test problems with known solutions.

Completely different approach, based on so called *price-consistent formulation*, was proposed in [6]. It is based on the idea that the All-KKT system can be reduced to have only one copy of the NCP constraint, since all leaders play with the same followers. Then we can assume that the multipliers on identical joint constraints are the same, hence the price consistency. In fact, we get a three level game with a new set of players, by Leyffer and Munson called independent system operators, that choose the multipliers on the constraints. For illustration, we present the example taken from [6].

Example 3

Consider a game

$$\text{minimize}_{x_1} x_1^2 + a x_1 x_2 \text{ subject to } x_1 + x_2 = c$$
$$\text{minimize}_{x_2} x_2^2 + b x_1 x_2 \text{ subject to } x_1 + x_2 = c,$$

where a, b and c are parameters. The solution of this EPEC can be computed by solving

$$\begin{aligned} &2x_1 + a x_2 + \mu = 0 \\ &2x_2 + b x_1 + \nu = 0 \\ &x_1 + x_2 = c. \end{aligned}$$

One can show that every point $(x_1, c - x_1)$ is an equilibrium point. However, the solutions of the price-consistent version of our EPEC can be found as solutions to

$$\begin{aligned} &2x_1 + a x_2 + \mu = 0 \\ &2x_2 + b x_1 + \mu = 0 \\ &x_1 + x_2 = c \end{aligned}$$

and it admits

1) a unique equilibrium if $a + b \neq 4$.

2) An infinite number of equilibria $(x_1, c - x_1, 2c)$ when $a = b = 2$.

3) an infinite number of equilibria $(x_1, c - x_1, 2x_1 - a x_1)$ when $a + b = 4, a \neq b$ and $c = 0$.

4) no equilibrium when $a + b = 4, a \neq b$ and $c \neq 0$.

It is important to notice that the solutions to price-consistent version of EPEC form just a subset of the set of solutions to original EPEC. For the full description of price-consistent method we refer the reader to [6].

Note, that all mentioned solution techniques for EPECs rely on efficient MPEC solvers. A solver developed specifically for EPECs still remain an important open research question.

5 Cooperative Equilibrium Problems with Equilibrium Constraints

Another crucial question in oligopoly theory is whether and under which conditions profit-maximizing firm will agree to cooperate with other market players. This, of course, depends on the number of times the market situation is repeated. We can naturally introduce this momentum also to our bilevel structure; the cooperation can take place on upper level or on lower level or on both levels, though separately. In order to make the situation reasonable to study, we consider the game to be repeated infinitely often (to ensure that firms can make the agreement in the belief that it will be sustained by self interest).

In this section let us consider just the possibility that the leaders cooperate in their decision making in order to leverage (supposedly illegally) their advantageous position on the market. A good understanding of this unwanted situation may be useful, e.g., to uncover, using mathematical tools, a market behavior that is in variance of market competition rules. Since for this type of EPECs the numerical approach is well developed, based on the results of numerical experiments, see subsection 5.2, we can derive some conclusions towards this end. This subsection is fully based on recent papers [10] and [4].

5.1 Mathematical formulation

Let the lower-level problem be defined as in subsection 4.1. Again, suppose that ω is closed and let $\varphi_i[\mathbb{R}^{(n+m)l} \to \mathbb{R}], i = 1, 2, \ldots, n$, be the individual objective of the ith leader. Taking a closed convex cone $K \subset \mathbb{R}^n$, an ordering of $\mathbb{R}^n$ is then specified via $z_1 \prec z_2 \Leftrightarrow z_1 - z_2 \in K$. Note that by taking $K = \mathbb{R}^n_-$ we arrive at the standard weak Pareto optimality.

A strategy pair $(\hat{x}, \hat{y}) \in \omega \times \Omega$ is declared to be a *weak Pareto solution* of EPCC if

$$0 \in F(\hat{x}, \hat{y}) + N_\Omega(\hat{y})$$

and if there is a neighborhood $\mathcal{U}$ of $(\hat{x}, \hat{y})$ such that the conditions

$$\begin{array}{l} 0 \in F(x, y) + N_\Omega(y) \\ \varphi(x, y) - \varphi(\hat{x}, \hat{y}) \in riK \end{array}$$

are inconsistent for all $(x, y) \in \mathcal{U} \cap (\omega \times \Omega)$.

The corresponding *multiobjective optimization* problem is now defined as follows:

$$\begin{array}{cl} \underset{K}{\text{minimize}} & \varphi(x, y) \\ \text{subject to} & 0 \in F(x, y) + N_\Omega(y) \\ & x \in \omega. \end{array} \tag{15}$$

5.2 Numerical approach and results

In this section we propose and describe a numerical method to solve the EPCCs under consideration based on the *implicit programming approach*, cf. [7]. One can employ a nonsmooth multiobjective optimization method, e.g., the online multiobjective optimization software WWW-NIMBUS 4.1. For details about NIMBUS we refer to the web page `http://nimbus.mit.jyu.fi/N4_1`.

Let us first "globalize" assumption (A1)' as follows.

(A1)" S is single-valued and locally Lipschitzian on an open set containing ω.

Under the assumption (A1)" we readily observe that $(\hat{x}, \hat{y}) \in \omega \times \Omega$ is a *weak Pareto* solution of EPCC whenever $\hat{y} = S(\hat{x})$ and there is a neighborhood $\mathcal{U}$ of $(\hat{x}, S(\hat{x}))$ such that (with $\Theta(x) = \varphi(x, S(x))$) the relation

$$\Theta(x) - \Theta(\hat{x}) \in riK,$$

does not hold for any $(x, S(x)) \in \mathcal{U} \cap (\omega \times \Omega)$. We face a new game only among leaders without any hierarchical structure in the form of a multiobjective optimization problem.

NIMBUS has to be provided with an oracle in Fortran 77 code which is able to compute the function values of each leader's objective and the matrix of subgradients $\left[\hat{\xi}_1, \ldots, \hat{\xi}_n\right]$, where $\hat{\xi}_i$ is an arbitrary element from the Clarke subdifferential $\bar{\partial}\Theta^i(x),\ i = 1, \ldots, n$, to compute it we use the technique of adjoint equations, see [10]. The followers' strategies for the given leaders' strategies can be computed by any existing method for the solution of MCP; we used the method based on the sequential quadratic programming code NLPQL due to Schittkowski.

WWW-NIMBUS 4.1 works as follows. The user must specify the starting point of the procedure. NIMBUS then computes a solution to the considered EPCC which we call *initial.* This point is a projection of the starting point onto the set of effective points. Since it is rarely satisfactory, the user can choose which of the function values should be decreased from the current level and which of the functions are less important. In NIMBUS, this process is called "classification". After submitting a new classification, NIMBUS provides a new optimal solution.

Let us have $n + m$ firms, producing a homogeneous product; $x_i \in \mathbb{R}, i = 1, 2, \ldots, n$, be the production of the ith leader, $y_j \in \mathbb{R}_+, j = 1, 2, \ldots, m$, be the production of the jth follower. The vector x of the leaders' productions must belong to some closed subset ω of $\mathbb{R}^n$. Let $T = \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j$ be the overall production, and $p : \operatorname{int} \mathbb{R}_+ \to \operatorname{int} \mathbb{R}_+$ be the so called *inverse demand curve* which assigns T the price at which consumers are willing to purchase. The objectives of the leaders can now be written in the form

$$\varphi^i(x, y) := c^i(x_i) - x_i p(T),\ i = 1, 2, \ldots, n,$$

and similarly the objectives of the followers attain the form

$$f^j(x,y) := c^{n+j}(y_j) - y_j p(T),\ \ j = 1,2,\dots,m,$$

where the functions $c^k[\mathbb{R}_+ \to \mathbb{R}_+], k = 1,\dots,n{+}m$ represent the production costs. In order to use the implicit programming approach we need to guarantee the validity of assumption (A1)'. To fulfill the assumptions of [10, Theorem 4.2], we usually require that

(i) the functions $c^{n+j}, j = 1,2,\dots,m$, are convex and twice continuously differentiable;

(ii) p is twice continuously differentiable and strictly convex on $\operatorname{int}\mathbb{R}_+$;

(iii) $\vartheta p(\vartheta)$ is a concave function of ϑ;

(iv) ω does not contain the zero vector.

Then for all $j = 1,2,\dots,m$ the objective of the jth follower is convex with respect to y_j. We have

$$F(x,y) = \begin{bmatrix} \nabla c^{n+1}(y_1) - p(T) - y_1 \nabla p(T) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \nabla c^{n+m}(y_m) - p(T) - y_m \nabla p(T) \end{bmatrix}$$

and, as proved in [12, Lemma 12.2], the respective partial Jacobian $\nabla_y F(x,y)$ is positively definite at each feasible pair (x,y). This implies that assumption (A1)' holds true. It follows that, at an optimal strategy pair $(\hat{x},\hat{y})$, the optimality conditions [10, Theorem 4.2] are satisfied.

We can see that these assumptions are reasonable. The only slightly problematic to guarantee ex ante may be assumption (iv), however, the situation when the leaders choose not to produce anything at all, while at least one follower will be supplying the market with goods, indeed does not make much sense.

In the following we present an academic example of realistic oligopolistic market form and computed results in cases of EPCC structures.

Example 4

Let all the production cost functions be in the form

$$c^i(x_i) = b_i x_i + \frac{\beta_i}{1+\beta_i} K_i^{-\frac{1}{\beta_i}} x_i^{\frac{1+\beta_i}{\beta_i}},$$

where b_i, K_i and $\beta_i,\ i = 1,\dots,n+m$, are given positive parameters. Further, let

$$p(T) = 5000^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}},$$

with a positive parameter γ termed *demand elasticity.*

Table 1: Parameter specification for the production costs

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
b_i	10	8	6	4	2
K_i	5	5	5	5	5
β_i	1.2	1.1	1.0	0.9	0.8

Table 2: Productions and profits - Stackelberg games with parameter settings

Stackelberg	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
$\gamma = 1.0$					
Production	55.5483	50.1342	50.5040	47.8997	42.9768
Profit	343.3453	400.008	463.9979	498.3845	502.6867
$p(T) = 20.2378$					
$\gamma = 1.3$					
Production	24.1420	27.6825	32.0356	33.5535	32.4854
Profit	68.1356	120.8576	180.8255	232.2085	267.6123
$p(T) = 14.8481$					
$\gamma = 1.7$					
Production	4.7536	12.0546	18.5392	22.7133	24.3708
Profit	3.1612	24.3768	61.8105	105.4377	145.4737
$p(T) = 11.1880$					

Each production cost function is convex and twice continuously differentiable. The inverse demand curve is twice continuously differentiable on $\operatorname{int} \mathbb{R}_+$, strictly decreasing and convex. We also observe that the so called *industry revenue curve*

$$Tp(T) = 5000^{\frac{1}{\gamma}} T^{\frac{\gamma-1}{\gamma}}$$

is concave on $\operatorname{int} \mathbb{R}_+$ for $\gamma \geq 1$. We also assume that each leader on the market has to produce at least a given positive production quantity. Hence we have fulfilled all the above assumptions (i)-(iv).

We perform numerical tests for $n+m = 5, \gamma \in [1, 2]$. For the parameters of the production cost function given by Table 1, Table 2 shows the productions and profits of all the firms for different values of parameter γ in the *Stackelberg case*, when Firm 1 is the only leader. One can object that the results for higher values of γ show that Firm 1 is not the strongest firm on the market. Recall, for the firm to become the market leader it needs to have some temporal advantage over the followers. This advantage may not be connected to market power but, e.g., to the level of acquired information. So we do find it realistic from an economic point of view and interesting to investigate.

For each setting of parameter γ in Tables 3, 4 and 5, we present three possible outcomes the *EPCC case*, when Firm 2 becomes the second leader. In each case, the first section corresponds to the initial solution given by NIMBUS. We believe that this result reflects

Table 3: Productions and profits - games with two leaders, $\gamma = 1.0$

EPCC, $\gamma = 1.0$	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Production	45.2558	42.5467	52.6768	49.6384	44.3219
Profit	357.8634	410.9407	529.9167	558.8654	555.3318
$p(T) = 21.3275$					
Production	50.8043	33.9370	53.0478	49.9381	44.5561
Profit	394.2186	357.6356	542.1837	570.0930	565.0856
$p(T) = 21.5254$					
Production	36.5470	50.1959	52.8052	49.7420	44.4028
Profit	311.8783	458.3726	534.1281	562.7208	558.6816
$p(T) = 21.3956$					

Table 4: Productions and profits - games with two leaders, $\gamma = 1.3$

EPCC, $\gamma = 1.3$	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Production	21.9186	20.9696	33.1310	34.3933	33.1208
Profit	76.3425	113.7331	200.0776	251.5930	285.7973
$p(T) = 15.3521$					
Production	23.9655	16.2282	33.4615	34.6475	33.3143
Profit	83.8358	97.1109	206.3064	257.8316	291.6324
$p(T) = 15.5116$					
Production	17.7795	26.5980	32.9483	34.2529	33.0142
Profit	65.7046	129.5801	196.7223	248.2261	282.6449
$p(T) = 15.2655$					

the power proportion of both leaders in the best way. The second and third sections represent situations when the contract between both leaders is more beneficial for Firm 1 and Firm 2, respectively.

We can see that except for $\gamma = 1.0$ (first section of Table 3), when both leaders improved their profits, the stronger leader, Firm 2, has to sacrifice a part of its profits to the benefit of Firm 1. We can also see that with increasing value of γ, Firm 1 grows weaker compared to Firm 2. So one can expect that the bigger the power difference between leaders the more the stronger one has to sacrifice.

For $\gamma = 1.0$, the differences between sums of the production quantities and profits of Firm 1 and Firm 2 in each situation do not exceed 5%. For $\gamma = 1.3$, the differences reach 10% and for $\gamma = 1.7$ the difference between sum of the profits of leaders reaches 50%. We can conclude that the stability of results of leaders depends heavily on the difference between their market power. Notice that all three remaining firms significantly increased their profits, and in each case, their production quantities are robust with respect to changes in contract between the leaders. We can also deduce that the more beneficial contract for the stronger leader the lower is the market price. However, it always exceeds the market price in Stackelberg game.

Table 5: Productions and profits - games with two leaders, $\gamma = 1.7$

EPCC, $\gamma = 1.7$	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Production	5.4294	6.5211	19.1728	23.1649	24.6961
Profit	4.8244	18.2977	68.1691	112.8277	153.1178
$p(T) = 11.4727$					
Production	5.9208	4.5826	19.3618	23.2992	24.7913
Profit	5.5272	14.1033	70.1925	115.1579	155.5186
$p(T) = 11.5615$					
Production	2.7593	12.5864	18.7298	22.8493	24.4682
Profit	2.5919	25.9189	63.6575	107.5954	147.7105
$p(T) = 11.2717$					

Table 6: Productions and profits - modified parameter $b_1 = 2$

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Stackelberg					
Production	99.5329	44.3804	45.8893	44.2806	40.2357
Profit	958.6347	284.6830	350.5039	393.2799	410.5319
$p(T) = 18.2270$					
EPCC, $\gamma = 1.0$					
Production	62.8288	49.1867	49.7383	47.2930	42.4805
Profit	840.8600	378.3762	442.9064	478.9642	485.6284
$p(T) = 19.8786$					
EPCC, $\gamma = 1.0$					
Production	88.8892	46.7669	47.7940	45.7640	33.7368
Profit	978.8980	328.1489	393.6102	433.3945	410.9734
$p(T) = 19.0150$					

We finish this section by modifying the parameter specifications from Table 1 and setting $b_1 = 2$ (instead of 10) to show the results of the situation when two strongest subjects on the market become cooperative leaders. For these data, the first section of Table 6 shows the productions and profits of all firms for $\gamma = 1.0$ when Firm 1 is the only market leader. We present the results when Firm 5 becomes the second leader in the second and the third section of Table 6. The former result again corresponds to the initial result given by NIMBUS and the latter one represents the situation under a contract that would be beneficial for both leaders.

6 Conclusion

In this bachelor thesis we investigated the complex oligopolistic market structures. The generalization of Stackelberg problem that introduces the equilibrium structure on the upper level problem has nowadays important practical application. As in the some western countries (e.g., UK, Australia and several US states) the electricity markets are already deregulated, for the local authorities is it crucially important to have a theoretical based numerical concept to keep the power distribution through the electricity system economically efficient.

In the first part of the thesis we discussed the “Cambridge model” of deregulated electricity markets. Then we presented the generalized mathematical formulation of this problem in terms of a new class of equilibrium problems. On several examples we discussed the theoretical difficulties generically connected to these problems and described several numerical approaches that can be already found in published literature. In the last part of the thesis we discussed the incidences of the cooperative behavior of leaders within this bilevel game and we presented our own numerical results.

I am convinced that such far reaching problems as EPECs will arise very soon also in the Czech Republic (after the privatization of power supply sector) and then it will be extremely useful to be aware of all suitable mathematical tools.

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Project of Bachelor Thesis

Schedule for the bachelor exam: **summer semester 2005/2006**
Author of the bachelor thesis: **Mgr. Michal Červinka**
Supervisor of the bachelor thesis: **Ing. Ivo Koubek**

Theme: **Oligopolistic Markets in Terms of Equilibrium Problems with Equilibrium Constraints**

Goals of the thesis:

In 2001 the brand-new concept of solving a situation on the deregulated electricity markets under an independent system operator (ISO) regime was presented. On this markets there are several energy producers supplying the market in the position of the market leaders and there is also the ISO possing as the single market follower. This game can be mathematically represented as a problem from the newly-emerged class of mathematical problems called Equilibrium Problems with Equilibrium Constraits (EPECs).

This bachelor thesis should serve as a state-of-the-art overview i.e. the goal of this thesis is to map the existing literature on EPECs. The mathematical formulation of the deregulated electricity market problem under ISO regime will be presented along with the general formulation of EPECs. Author will show the problematic intrinsic features of EPECs (e.g. lack of convexity, huge computational complexity) in terms of simple examples. In the end of the thesis the existing computational approaches will be discussed.

Synopsis:

1. Preliminaries and mathematical tools
2. Deregulated electricity markets under an independent system operator (ISO) regime
3. Mathematical formulation of EPECs
4. Academic examples illustrating the problematic features of EPECs
5. Analysis of the existing computational approaches

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In Prague on ....

Signature of the supervisor   Signature of the author