Charles University in Prague
Faculty of Social Science
Institute of Economic Studies

Bachelor Thesis

2007 **Boril Šopov**

CHARLES UNIVERSITY IN PRAGUE
FACULTY OF SOCIAL SCIENCE
INSTITUTE OF ECONOMIC STUDIES

Bachelor Thesis

Intertemporal Redistribution: Strategic Aspects

Author: **Boril Šopov**
Supervisor: **PhDr. Martin Gregor, PhD.**
Academic Year: **2006/2007**

Prohlášení

Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a použil pouze uvedené prameny a literaturu.

V Praze dne 18.6.2007 Boril Šopov

Acknowledgment

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. I want to thank the Institute of Economic Studies for giving me the change to study there.

I am deeply indebted to my supervisor PhDr. Martin Gregor, PhD. whose help, stimulating suggestions and encouragement helped me in all the time of research for and writing of this thesis. I want to thank to my friends Bc. Pavel Dvořák and Kat Kinsella MSc. for valuable hints.

Especially, I would like to give my special thanks to my parents whose patient support enabled me to complete this work.

Abstract

This paper studies the incentives which are helping the social security system be sustained over time, while trying to answer the question, what politico-economic forces are behind such a massive redistribution. A positive model is built combining microeconomic foundations of individuals' preferences with public choice probabilistic voting concept. Such settings allow us to model preferences of young and old individuals to the level of intertemporal redistribution. Moreover, we can examine the influence of changes in longevity on the voting model outcome.

We conclude that the increase in longevity increases the level of redistribution.

JEL classification: C61 D72 J11

Keywords: intertemporal redistribution, longevity, probabilistic voting, social security

Abstrakt

Tato práce studuje podněty, které pomáhají zachovat současný penzijní systém v čase a zároveň se pokouší zodpovědět otázku, jaké politicko-ekonomické síly jsou v pozadí takovéhoto masivního přerozdělování. Sestavili jsme positivní model kombinující mikroekonomické základy preferencí jedince se stochastickým hlasováním z oblasti Veřejné volby. Takovéto prostředí nám umožňuje modelovat preference mladého i starého jednotlivce vůči úrovni mezičasové redistribuce. Navíc, nám umožňuje zkoumat vliv změn v dlouhověkosti na výsledek volebního modelu.

Usuzujeme, že s prodlužováním délky života poroste úroveň přerozdělování.

JEL klasifikace: C61 D72 J11

Klíčová slova: mezičasové přerozdělování, dlouhověkost, stochastické hlasování, penzijní systém

Contents

1 Introduction **2**

2 Survey of Literature **4**

2.1 Economic Environment 4

2.2 Political Environment 7

2.3 Demography and Reforms 10

2.4 Summary 13

3 Model of Social Security **14**

3.1 Economic Nature 14

3.2 Preferences 18

3.2.1 Utilitarian Government 18

3.2.2 Young Individuals 20

3.2.3 Old Individuals 29

3.2.4 Increases in Longevity 34

3.2.5 Concluding Remarks 36

3.3 Electorate Competition 37

3.3.1 Preferred Policy 37

3.3.2 Probabilistic Voting 38

3.3.3 Application and Interpretation 42

4 Conclusion **46**

A Mathematical Background **48**

Chapter 1

Introduction

Everyone wants to live at the expense of the state, they forget that the state lives at the expense of everyone.

Frederic Bastiat

Nowadays, all developed countries provide some kind of social security. This paper studies the incentives helping the social security system be sustained over time. We try to answer the question what politico-economic forces are behind such a massive redistribution. Therefore, we review relevant approaches from which we select a few to build the model on. The model is predominantly based on Andersen (2006, 2005) while combining the key aspects of heterogeneous population (inspired by Razin and Sadka (2004)), within cohort redistribution and reduced time horizon with probabilistic voting.

To the author's knowledge, this combination is unique and owing to this, we are able to capture many empirically observed effects. Furthermore, due to the heterogeneous population, we can easily model the preferences of individuals, and thus we naturally face the political competition between the rich and the young individuals. Since the social security redistributes in an intertemporal manner, the old individuals play a significant role. Although not numerous yet not divided into rich or poor cohort, the old individuals swim in the electorate competition as one man and tilt the outcome in their favour. When the increases in longevity figure in, the old cohort becomes even stronger. They influences the size of the social security system and the outcome level of redistribution rises. In addition, we try to identify problems that we expect to be caused by the increasing longevity.

The paper proceeds in the following manner; initially we define a simple economic environment, an individuals' utility function from consumption and a government utility function. Optimising the given functions, we receive preferences towards the level of redistribution and a basic mechanism for governmental control for increases in longevity. Owing to the heterogeneity of the population, the model identifies young individual indifferent towards changes in the level of redistribution, in other words the pivotal voter. The pivotal voter plays a key role; she splits the young group into two cohorts: the poor young and the rich young cohort.

Furthermore, having derived individuals preferences, we employ a slightly adjusted probabilistic voting model by Persson and Tabellini (2000). This model allows us to aggregate preferences of three distinct groups and gives a clear answer to what politico-economic forces are behind such a massive redistribution. The answer is the within cohort redistribution; the poor young individuals from a coalition with the old individuals. In addition, the increases in longevity increase the size of the old cohort, and thus strengthen this effect.

The paper is organised as follows Chapter 2 surveys the literature, Chapter 3 builds the model of social security and Chapter 4 concludes. For reader's convenience, there is the mathematical background presented in Appendix A.

Chapter 2

Survey of Literature

This chapter reviews the literature that aspires to explain and model the politico-economic forces that give rise to social security. Moreover it discusses why social security is sustained in the modern world and it prepares a background for the model of social security.

The survey is predominantly based upon studies of Galosso and Profeta (2002) and Mulligan and Sala-i-Martin (1999b,c). The first study deals with political processes of social security and with different economic explanations[1], whereas the second compares the empirical facts with conclusions suggested by various models and also adds conclusions about possible reforms of social security systems while facing demographic changes. In a nutshell, Galosso and Profeta (2002) assume an OLG model with three generations alive, closed economy, nonnegative population growth and pension benefits financed with a flat tax.[2]

2.1 Economic Environment

This section deals with different economic settings, it shows how the existence of social security can be explained by using diverse environments and how

[1]Galosso and Profeta introduce an environment, where they test various economical factors.

[2]See Galosso and Profeta (2002) for more details.

these environments affect the shape of social security. Galosso and Profeta (2002) review four mainly used approaches: dynamic inefficiency, crowding out effect, reduced time horizon and within cohort redistribution.

Dynamic Inefficiency The first approach explaining the existence of social security is the assumption of dynamic inefficiency. If the population growth rate is higher than the real rate of return from capital accumulation, then the economy is dynamically inefficient. The living young generation favours a social security system as they will obtain transfer[3] from the more numerous future young generation.[4] This notion makes young cohorts prefer the existence of social security, since they benefit from the economy's inefficiency. According to Galosso and Profeta (2002), this approach is not sufficient, as there are both economic and theoretical problems. It turns out that the assumption of nonnegative population growth rate is quite strong. In addition, the population growth rate converges to zero in many countries (mainly in Europe).

Crowding Out Effect Through the crowding out effect it is possible to analyse further the impact of the existence of social security systems, yet it stays out of this paper's interest.

The main idea is that intergenerational redistribution[5] may crowd out capital. By general mechanisms this increases the rate of return to capital and decreases the real wage, thus redistributing from workers to asset holders. Although there are a few studies on this notion, the practical relevance remains to be deeply analysed.

Reduced Time Horizon Another approach uses the reduction of the individual's time horizon. The notion is quite simple and this time we assume

[3]The transfer is higher than their contribution.

[4]There is an implicit assumption of irreversibility, i.e. they obtain what has been promised.

[5]Mulligan and Sala-i-Martin (1999a) mention social security or simply creation of public debt.

a dynamically efficient economy with three generations (young, middle-aged and old generation). The middle-aged individuals prefer positive redistribution, because they take their previous contributions as sunk costs (Galosso and Profeta (2002)). Consequently, the implicit rate of return to social security will be higher, as they consider only future contributions in a reduced time horizon. Therefore, two generations support social security, which is enough to put such system into practice.

Within Cohort Redistribution Within cohort redistribution, as suggested by Persson and Tabellini (2000), may be another reason why social security system exists. Social security is financed by some kind of flat income tax in, virtually, every country (and in every model), yet the benefits are usually paid *per capita*, which makes social security progressive. Young, low-income cohorts may benefit from redistribution from high-income individuals. This side effect makes social security, for low-income cohorts, more profitable than capital accumulation (Galosso and Profeta (2002)). The implication is that higher inequality would lead to higher redistribution in social security.[6]

Altruism Altruism can be taken into account in order to explain why young individuals favour social security (Galosso and Profeta (2002)). Young individuals with altruistic preferences support the social security system, relying on the future young altruistic generation. Galosso and Profeta point out that if an individual has not saved enough for his old-age consumption, the future young generation will be willing to provide them an old-age transfer. This behaviour would lead to intentional 'under-saving' and as such shifting resources to the future.

Interestingly, this social security system would be enacted under the unanimity rule[7] (Galosso and Profeta (2002)). Tabellini (2000) introduced a model

[6] Galosso and Profeta (2002) suggest that the empirical relevance is not clear.

[7] Which would lead to the improvement of allocation, as old cohort will not support any less beneficial programme, that would make them worse off.

combining weakly altruistic preferences with intergenerational redistribution. Although young individuals' weak altruism is not enough to support transfer to the elderly, the intergenerational redistribution effect causes young low-income individuals to support a positive level of redistribution as they benefit from the transfer that their parents obtain.

2.2 Political Environment

Galosso and Profeta (2002) study three broad classes of political arrangements that are used to aggregate the population's preferences into one outcome. Those are: majority voting, veto-power or constitutional rules, and interest-groups or lobbying. Lastly, there is one more type of institution that is quite widely used; probabilistic voting.

Majority Voting To apply majority voting to social security, we need to set up one parameter to vote for.[8] Usually, this parameter is a tax rate that directly affects the level of intergenerational redistribution. Another issue to be solved is the obligation that future generations keep the social security system unchanged. Galosso and Profeta (2002) reviewed early once-and-for-all models and considered them as unrealistic. Later studies came up with implicit contracts between generations (Galosso and Profeta (2002)).

Majority voting without any extensions, in the context of social security, would lead to no redistribution. There will not be a majority in a ballot whether to sustain the social security (redistribution from workers to retirees) or not, because over 50% of population would not benefit from it, as in the real world the median voter is not a retiree (Mulligan and Sala-i-Martin (1999b)).

Veto Power Sjoblom (1985) significantly proceeded to expand on previous studies and introduced a model that implicitly bound generations. Every pe-

[8] This notion is quite crucial, as in a model with more than one parameter (dimension), there may not exist the Condorcet winning policy.

riod there is a vote over existing social security system, he does not consider preferences of the old generation, as those are not relevant in majority voting.[9] A young individual either supports the system, or not. The mechanism uses a punishment for not supporting the system; she will not be provided with any transfer when old. Hence, even a self-interested individual will favour to leave the existing social security system unchanged. The notion of punishing those who do not support the system has been criticised, namely because it brings a high degree of indeterminacy, moreover it does not bring any sharp implications (Galosso and Profeta (2002)).

Discussing social security we find out that implementation of such system usually requires wider support than just the majority. To bargain for it, the institution of veto power was introduced (Hansson and Stuart (1989)). Giving the elderly the power to block any changes, that would make them worse off, leads to a Pareto-improving equilibrium. This is because young voters will not propose any changes that could possibly be rejected. Another application of this notion allows the electorate to choose between majority and veto power voting.[10] It suggests that even the possibility of using veto power is enough to achieve the Pareto-effective equilibrium. On the contrary, Galosso and Profeta (2002) point out that usually there is no constitutional veto power in social security policy decisions.

Interest-group The interest-group models (or lobbying) bring new information about the way the old cohort put through social security systems while usually being less numerous than other groups. According to Galosso and Profeta (2002), interest-group models use two approaches: influence function (two groups apply political pressure to policy-makers through an influence function) and support function models (government maximises a political support function).[11] The first to deal with the interest-group

[9]For two reasons, they do not form a majority and they obviously favour social security.

[10]Azariadis and Galosso (2002) compared outcomes from majority and veto power voting games.

[11]Support function is sometimes perceived as a special case of the influence function.

model was Becker (1985). He stressed the influence of pressure groups rather than the voting process itself, an approach later used by Mulligan and Sala-i-Martin (1999a). They assume two groups of identical individuals. Each members of each group is dedicated to political activity in order to get transfer from the other group; these groups represent young and old individuals.

Moreover, there are many determinants that may change the whole outcome. Those are: size of groups, level of dedication (including financial and time cost of campaign or lobbying) and the free-rider problem. The last one is becoming quite important in the case of larger groups, if necessary it can be solved by imposing a policy to align members' interest with those of the group (Galosso and Profeta (2002)).

These models assume the old generation to have greater influence on policy-makers or government, as they have got more free time or are more politically organised. The outcome is again positive rate of redistribution.

Probabilistic Voting This voting setting moves the models closer to the reality. Unlike the median voter approach (as criticised by Gonzales-Eiras and Niepelt (2005)), probabilistic voting does not have that many difficulties when dealing with more than one dimension.[12] In the real world, while deciding, individuals do not consider only proposed policy (i.e. social security system), but also bear in mind other dimensions, such as party ideologies, personal characteristics of a party leader, popularity of candidates amongst electorate, etc. Persson and Tabellini (2000) refers to this dimension simply as 'ideology'. This approach offers an explanation for outcomes from votes on social security that might not usually reflect the result suggested by median voter concept. As parties maximise their probability of being elected, they may propose a policy close to the one that corresponds with median voter.

Persson and Tabellini (2000) suggest the following timing of events: candidates/parties propose their policies knowing the distributions of the 'ideol-

[12]There may not exist the Condorcet winning policy in the case with more dimensions.

ogy' parameter[13], then the 'ideology' factors are realised, elections are held and finally, the winning candidate/party implements its policy.

There is a possibility of extending the probabilistic voting model, as suggested by Persson and Tabellini (2000), by adding a lobbying feature. By the time the candidates/parties have proposed their policies and before the voters have realised their 'ideology' factors, the interest groups come into the process. They can help the candidates/parties finance their campaigns or even take part in them, so that they influence voters and increase the chance that their preferred party will win.

This extension brings similar implications as with interest-group models. The old individuals are supposed to have more free time to dedicate to campaigning and the high income cohorts are able to finance the campaign.

2.3 Demography and Reforms

The survey reviews political and economic forces to explain the existence of the social security system so far. At the moment, we take a closer look at demographic aspect of the problem, which may affect the sustainability of the system as such. This approach is sometimes omitted. Even such a thorough study as presented by Mulligan and Sala-i-Martin (1999b,c) left out this perspective. This limits their analysis, both at positive and at normative level (Profeta (2000)).

There are two demographic factors that have direct negative impact on social security systems: decrease in fertility and increase in life expectancy. Consequently, these factors cause increases in dependancy ratio and aging of population and they both may change the age structure of population. As a result, the social security system is likely to end up in financial turmoil. Previously mentioned problems can be mitigated by adjusting the parameters of the system[14] and/or by changing the financing structure of the social

[13]The candidates/parties do not know the actual value of this parameter of a single individual, but they are aware of the distribution amongst the groups.

[14]Such as: the retirement age, the rate of contributions, the amount of benefits etc.

security system. These changes could represent reforms to a partially or fully funded social security systems.

Often, it is quite difficult to find the wide political consensus to apply such changes and it is even more difficult to reform an existing social security system. Mulligan and Sala-i-Martin (1999b) made an interesting remark, emphasising that there have not been any fully funded social security systems over significant period of time, even though a few fully funded systems have existed for short periods of time.

Sinn and Uebelmesser (2002) study the impact of demographic crisis in Germany on the feasibility of social security reforms.[15] They identify the final year[16] when there will be a majority who would benefit from the reform, to be exact: the year when the median age will be lower than the indifferent age. Sinn and Uebelmesser (2002) compare rates of return from the social security system (in the case of implementing reforms and in the case of not implementing them) for individuals from different age groups and find an age group indifferent to reform. After that year there will not be any political power to enact such reforms. Hence the current social security would end up in a financial turmoil, as it would not be able to guarantee to pay the 'promised' benefits.

Uebelmesser (2003) expands on the previous paper and applies the approach in three European countries (France, Germany and Italy) where various social security reforms have taken place recently. She identifies, moreover, the latest time when there is still a majority to support any reform of social security. She concludes that in reality the decline in the support can be postponed and might not be that clear, as not all individuals are aware of the situation of their cohort nor the impact of any reforms on their benefits. Nevertheless, these problems are likely to arise in the future and the need

[15] They stress that Germany applied several reductions to defined benefits of social security, most recently in 2001. These reductions were compensated on fiscal bases.

[16] The calculations were based on data from the German Council of Economic Advisors to the Federal Ministry of Economics and Research; see Sinn and Uebelmesser (2002) for details.

for reforms would not be balanced by any power to put them through.

Galosso and Profeta (2003) analyse the nature of such reforms in OECD countries. Their study focuses on demography and reforms which have taken place in many OECD countries. Linking back to what has been said above, most of these reforms have been parametric. The study closely analyses the social security systems by simulating and comparing features in France, Germany, Italy, Spain, the UK and the US. They emphasise the fact that although demographic changes do affect the social security, it is also the political process that shapes the reforms.

Profeta (2002) contributes with empirical study, builds up a new data set and by using the OLS estimation he finds a strong relationship between the retirement policies and the social security. Furthermore, he points out that retirement policies do not sufficiently reflect tendency towards aging populations, and thus allow older populations to work less and enjoy longer length of retirement.

Both previously mentioned studies draw similar conclusion about the solution to the problem of aging or at least a recommendation to cushion it. The former study stresses that the increase in retirement age always decreases the size of the system and the latter emphasises that the retirement age could be increased[17] in order to distribute the costs of the population aging amongst generations.

In conclusion, the demographic aspects that affect social security systems, cannot be omitted. Not only can the demographic changes in population make the systems deficit, but also can block any future amendments and reforms.

[17]This adjustment does not necessarily have to be at the expense of the length of retirement, as the increase in life expectancy may outweigh it.

2.4 Summary

In this chapter, we reviewed literature on sustainment of social security and related topics. In the next chapter we present a model of social security that contributes to explain why there exist social security systems. The model was inspired by approaches and ideas reviewed in this chapter, namely: reduced time horizon, within cohort redistribution and probabilistic voting. Its conclusions are also in accordance with the section 2.3.

Chapter 3

Model of Social Security

The state is the great fictitious entity by which everyone seeks to live at the expense of everyone else.

Frederic Bastiat

3.1 Economic Nature

This section builds on three studies: Razin and Sadka (2004) and Andersen (2006, 2005). The model is inspired by the education parameter presented in Razin and Sadka (2004), yet it was replaced by original 'ability' parameter, a. This parameter a does not depend on the tax rate as in Razin and Sadka (2004), thus it does not distort the labour supply. This would come in useful later while finding the optimal tax rate. This model captures the effects of increases in longevity using the mechanism from Andersen (2006, 2005).

Productivity of Work – Ability Parameter There is a continuum of individuals with the innate ability parameter, $a^i \in \langle 0, 1\rangle$. An individual works and produces $(1 + a)$ units of effective labour. The ability parameter is distributed amongst the population by the cumulative distributive function denoted by $G(a)$, with the domain range $\langle 0, 1\rangle$ and the density function is continuous and denoted by $g = G'$. We assume this distribution to remain unchanged over time, i.e. all generations are able to produce equal effective labour. For later reference it is useful to define a^m as the individual with the median ability parameter a, and $\overline{a}$ as the individual with the mean ability parameter a:

$$a^m : \quad G(a^m) = 1/2; \qquad\qquad \overline{a} : \quad \overline{a} = \mathrm{E}(a).$$

Economy This paragraph builds on Razin and Sadka (2004). We assume an economy with access to an international capital market offering a risk-free asset with return r which is for simplicity assumed invariable throughout the time. We also assume the wage, w, to be invariable. The production function is linear in effective labour, L, and capital, K:

$$Y = wL + (1 + r)K,$$

where Y is gross output. As in Razin and Sadka (2004), the wage rate, w, and the gross rental price of capital, $1+r$, are determined by the marginal productivity conditions for factor prices ($w = \frac{\partial Y}{\partial L}$ and $1+r = \frac{\partial Y}{\partial K}$) and, importantly, are already substituted into the production function. The assumed linearity of the production function can arise as an equilibrium outcome through either international capital mobility or factor prices equalisation arising from goods' trade (Razin and Sadka (2004)).

Individuals We build a standard two generation OLG model of PAYG type, that is based on Andersen (2006, 2005). Individuals live two life-phases, one denoted Young and the other Old, the length of both periods are normalised to unity. We assume constant population, thus population growth equal to zero. By contrast, the longevity may change. The individuals live for the whole first period, while the second lasts β (≤ 1); the change in β is a change in longevity. In the first period, the individuals are not aware of β, the parameter β is known in the second live phase and the same for all individuals in the period. We assume the number of individuals in a life-phase to be normalised to unity. We denote the generation which is young in period t as generation t. When young, individuals work and enjoy a consumption stream $\frac{c_{1t}}{1}$, yielding utility:

$$u(c_{1t}) \qquad u' > 0, u'' \leq 0.$$

During the second period of life, the individuals of generation t are alive for β_{t+1} and they work a fraction of the period α_{t+1} ($\leq \beta_{t+1}$), we assume α to be set by the government, and thus the same for all individuals in a given period. The consumption in the second period is c_{2t+1} yielding a consumption stream of $\frac{c_{2t+1}}{\beta_{t+1}}$ and the utility is:

$$\beta_{t+1} u\left(\frac{c_{2t+1}}{\beta_{t+1}}\right).$$

Note that this specification implies that agents value long life time *ceteris paribus*, but the utility function implies a trade-off between longevity and consumption (Andersen (2006)).

Furthermore, we assume that the disutility of work in the first period is constant for all individuals and hence disregarded for simplification. In the second period, when old, the disutility of work is given:

$$\alpha_{t+1} v\left(\frac{\alpha_{t+1}}{\beta_{t+1}}\right) \qquad v' > 0, v'' > 0, \lim_{\alpha \to \beta} v' = \infty.$$

This makes the disutility convex and the last condition ensures that $\alpha < \beta$, i.e. that there is always some 'retirement' period. For later reference, it is useful to define the marginal disutility of work as old by:

$$\eta\left(\frac{\alpha_t}{\beta_t}\right) \equiv \frac{\partial}{\partial \alpha}\left(\alpha_t v\left(\frac{\alpha_t}{\beta_t}\right)\right) = v\left(\frac{\alpha_t}{\beta_t}\right) + \frac{\alpha_t}{\beta_t} v'\left(\frac{\alpha_t}{\beta_t}\right),$$

where $\eta' = 2v' + \frac{\alpha}{\beta} v'' > 0$, $\eta'' = 3v' + \frac{\alpha}{\beta} v''' \lesseqgtr 0$. Note that for $\frac{\alpha}{\beta}$ sufficiently large it follows that (given that $\lim_{\alpha \to \beta} v' = \infty$) $v''' > 0$.

Labour Supply As we assume individuals to work throughout their entire young phase and a part (α) of their old phase, the individual's labour supply when young is:

$$\ell_y^i = (1 + a^i),$$

and when old:

$$\ell_o^i = \alpha \ell_y^i,$$

where ℓ_y^i and ℓ_o^i are the effective labour supplies of a young individual and the effective labour supply of an old individual, respectively. As the distribution of the parameter a is invariant throughout the time, the effective labour supply of young and old individual reads:

$$L_y = \int_0^1 \ell_y^i(a)\mathrm{d}G, \qquad L_o = \alpha L_y.$$

The aggregate effective labour supply reads:

$$(1+\alpha)L_y = L_y + L_o = L(\alpha),$$

where $L(\alpha)$ is the total aggregate effective labour supply and $\int$ represents the Stieltjes integral. Note that the aggregate effective labour supply $L(\cdot)$ becomes a linear function of the parameter α.

Individual's Wealth We assume that an individual works for the entire young phase and a part (α) of her old phase. Hence, the present value of her wealth is in period t – when young:

$$\omega_t = (1-\tau)w\ell_y^i + \frac{(1-\tau)w\ell_o^i}{1+r} + \frac{b}{1+r},$$

rearranging terms and substituting $\ell_o^i = \alpha \ell_y^i$:

$$\omega_t = (1-\tau)(1+a^i)w\Big(1+\frac{\alpha}{1+r}\Big) + \frac{b}{1+r}, \tag{3.1}$$

where ω_t is a wealth of individual young in period t, τ is a income tax rate ($\tau \in \langle 0, 1\rangle$) and b is a lump sum *per capita* benefit paid when retiring. This setting makes the system progressive.

Hence the individual's consumption constraint reads:

$$c_{1t} + \frac{c_{2t+1}}{1+r} = \omega_t. \tag{3.2}$$

Government Budget Constraint The government pays every individual in her second period a lump sum benefit b. The benefits are financed by the income tax that is paid by all working individuals in the period t. Hence the benefits equal:

$$b = \tau w L(\alpha). \tag{3.3}$$

3.2 Preferences

In this setting we focus on preferences of government over the retirement parameter α and of individuals over the tax rate τ. The government sets parameter α as the utilitarian optimum. The individual's disutility of work, ensures that there is always some retirement period ($\alpha < \beta$).

3.2.1 Utilitarian Government

The utilitarian government maximises the utility function of the living two generations and also the discounted utility of all future generations. Before denoting the government utility function, we need an individual's direct utility function $V_t(c_{1t}, c_{2t+1}, \alpha_{t+1}, \beta_{t+1})$:

$$V_t(\cdot) = u(c_{1t}) + \frac{1}{1+\rho} E_t \left[\beta_{t+1} u\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big) - \alpha_{t+1} v\Big(\frac{\alpha_{t+1}}{\beta_{t+1}}\Big) \right], \tag{3.4}$$

where ρ is the subjective discount rate. Note that $V'_{c_1} > 0$, $V'_{c_2} > 0$, $V'_{\alpha} < 0$ and $V'_{\beta} \lesseqgtr 0$, *ceteris paribus*. Interestingly, due to the budget constraint the individual's utility may increase with the increasing parameter α. This may come up when the preset parameter α is lower than the individual's preferred one. Therefore, an increase in the parameter α may increase the utility of that individual, as the extra income yields the extra increase in consumption stream, and thus an extra increase in utility that outweighs the disutility of the extra work.

Similarly as in Andersen (2005), we define the utility of individuals alive in period t as $S(\cdot)$ by contrast without discounting the utility of any living generation:

$$S(\mathbf{x}_t) = S(c_{1t}, c_{2t}, \alpha_t, \beta_t) \equiv u(c_{1t}) + \beta_t u\Big(\frac{c_{2t}}{\beta_t}\Big) - \alpha_t v\Big(\frac{\alpha_t}{\beta_t}\Big).$$

Note that $S'_{c_1} > 0$, $S'_{c_2} > 0$, $S'_\alpha < 0$ and $S'_\beta \lesseqgtr 0$. The government utility function reflects the utility of the old in time t, but enjoyed when young in time $t-1$:

$$\Psi_t(\mathbf{x}_t) = (1+\theta)E_t\Big[u(\bar{c}_{1t-1}) + \sum_{j=0}^{\infty} \frac{S(\mathbf{x}_{t+j})}{(1+\theta)^j}\Big], \tag{3.5}$$

where θ is the discount rate assigned by government to the future generations' utility. To make the notion simple, we unwind the function Ψ:

$$\Psi_t(\mathbf{x}_t) = \underbrace{(1+\theta)u(\bar{c}_{1t-1}) + \beta_t u\Big(\frac{c_{2t}}{\beta_t}\Big) - \alpha_t v\Big(\frac{\alpha_t}{\beta_t}\Big)}_{\text{current old}} +$$

$$+ \underbrace{u(c_{1t}) + \frac{E_t\Big[\beta_{t+1} u(\frac{c_{2t+1}}{\beta_{t+1}}) - \alpha_{t+1} v\Big(\frac{\alpha_{t+1}}{\beta_{t+1}}\Big)\Big]}{1+\theta}}_{\text{current young}} + \dots$$

The aggregate resource constraint is:

$$\sum_{j=0}^{\infty} \Big(wL(\alpha_{t+j}) - c_{1t+j} - c_{2t+j}\Big)(1+r)^{-j} = 0, \tag{3.6}$$

where $wL(\alpha_{t+j})$ denotes the national wealth in period $t+j$. The government maximises the utility function, subject to the aggregate resource constraint:

$$\arg\max \Psi_t(\mathbf{x}_t)$$

subject to (3.6). The first order conditions[1] of the optima are:

$$u'(c_{1t}) = u'\Big(\frac{c_{2t}}{\beta_t}\Big) \quad \text{(within period consumption allocation)}, \tag{3.7}$$

$$u'\Big(\frac{c_{2t}}{\beta_t}\Big) = \frac{1+r}{1+\theta} E u'\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big) \quad \text{(across period consumption allocation)}, \tag{3.8}$$

[1]See Appendix 1 for more details.

$$u'\Big(\frac{c_{2t}}{\beta_t}\Big)wL_y = \eta\Big(\frac{\alpha_t}{\beta_t}\Big) \quad \text{(retirement age)}, \tag{3.9}$$

$$\eta\Big(\frac{\alpha_t}{\beta_t}\Big) = \frac{1+r}{1+\theta}E\eta(\frac{\alpha_{t+1}}{\beta_{t+1}}) \quad \text{(across period retirement allocation)}. \tag{3.10}$$

These four conditions give us the optimal allocation. The condition (3.7) gives the within period allocation; the weighted utility of the old and the young must be equal. Andersen (2005) points out that this condition is found in Bohn (1999)[2], and it shows that all living generations should participate equally in risk sharing. The across period consumption allocation is shown by the condition (3.8) and it holds in different periods; the marginal utility of the currently living old is proportional to the expected marginal utility of next generation of old individuals. This condition determines the risk sharing across generations. The condition (3.9) gives us the retirement age, thus giving the optimal parameter α_t^*. In addition, it gives the standard link between consumption and leisure.

3.2.2 Young Individuals

Recalling the direct utility function of a young individual (3.4), and the individual's budget constraint (3.1), we can maximise her utility:

$$\arg\max_{c_{1t},c_{2t+1},\alpha_{t+1}} V_t(c_{1t}, c_{2t+1}, \alpha_{t+1}, \beta_{t+1})$$

subject to (3.2): The first order conditions[3] to the individual's problem of maximising are:

$$\frac{1}{1+r}u'(c_{1t}) = \frac{1}{1+\rho}Eu'\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big), \tag{3.11}$$

$$Eu'\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big)\Big[(1-\tau)(1+a^i) + \tau L_y\Big]w = E\eta\Big(\frac{\alpha_{t+1}}{\beta_{t+1}}\Big). \tag{3.12}$$

[2]Bohn (1999) studies changes in life expectancy under uncertainty.

[3]See Appendix 2 for more details.

The first order conditions of the individual's problem of maximising differ from those of the government's. Mainly because an individual pursuits her own interests (neither concern for the others nor the future generations), due to different discounting the future utility and lastly due to the different budget constraint, which is specific amongst individuals. The condition (3.11) gives the individual's inter-period consumption allocation; the weighted marginal utility as young equals the weighted marginal utility as old. The condition (3.12) gives us the optimal retirement age for the individual.

Assuming α to be preset by the government, the individual's utility increases with the increases in c_{1t} and c_{2t+1} ($u' > 0$). As the tax rate τ does not distorts the effective labour supply, we can try to find an individual indifferent to changes in τ *ceteris paribus*, such individual becomes a pivotal voter. We recall the individual's budget constraint (3.1) and the government budget constraint (3.3), we can calculate the individual's preferred tax rate τ. Deriving (3.1) with respect to τ we receive:

$$\begin{aligned}\frac{\partial \omega_t}{\partial \tau} &= \frac{\partial}{\partial \tau}\Big[(1-\tau)(1+a^i)w\Big(1+\frac{\alpha_{t+1}}{1+r}\Big)+\frac{b}{1+r}\Big] = \\ &= \frac{\partial}{\partial \tau}\Big[(1-\tau)(1+a^i)w\Big(1+\frac{\alpha_{t+1}}{1+r}\Big)+\frac{\tau L w}{1+r}\Big] = \\ &= -(1+a^i)w\Big(1+\frac{\alpha_{t+1}}{1+r}\Big)+\frac{Lw}{1+r} \lesseqgtr 0\end{aligned}$$

Rearranging terms to make the equation clear:

$$(1+a^i)w\Big(1+\frac{\alpha_{t+1}}{1+r}\Big) \lesseqgtr \frac{Lw}{1+r}. \tag{3.13}$$

The equation (3.13) gives us the expected notion that the wealth of the 'net contributors' decreases with the tax rate τ and *vice versa*. Considering that $a^i \in \langle 0,1\rangle$, we see that $\frac{\partial \omega_t}{\partial \tau} \lesseqgtr 0$, supposing r to be rationally small[4], hence there exist $a^0 \in \langle 0,1\rangle$ so that $\frac{\partial \omega_t}{\partial \tau} = 0$, the individual with such ability parameter a^0 becomes indifferent to changes in the tax τ. Since the ω_t is linear in τ, it holds that:

$$\text{for } \forall a^i < a^0 : \frac{\partial \omega_t}{\partial \tau}(a^i) > 0, \quad \text{for } \forall a^i > a^0 : \frac{\partial \omega_t}{\partial \tau}(a^i) < 0.$$

[4] See the the discussion bellow the Proof of the Corollary 1 for more details.

Corollary 1 *If we denote P^Y as a set of poor young individuals, $P^Y := \{a^i : a^i < a^0\}$ and R^Y as a set of rich young individuals, $R^Y := \{a^i : a^i > a^0\}$ then*

(i) For $\forall a^i \in P^Y : \frac{\partial w^i}{\partial \tau} > 0$.

(ii) For $\forall a^i \in R^Y : \frac{\partial w^i}{\partial \tau} < 0$.

The poor young individuals prefer the highest tax rate τ as possible, which is $\tau = 1$, whereas the rich young individuals prefer the lowest tax rate τ as possible, which is $\tau = 0$. ♣

Proof We want to find such a^0 that $\frac{\partial w^o}{\partial \tau} = 0$. Rearranging terms and substituting $(1 + \alpha_{t+1}) \int_0^1 (1 + a)\mathrm{d}G$ for L:

$$(1 + a^i)(1 + r + \alpha_{t+1}) = (1 + \alpha_{t+1}) \int_0^1 (1 + a)\mathrm{d}G.$$

Consequently, we need the following three theorems, the first presented in Kalenda (2005), the second in Dresher (1981) (in Weisstein (2006)) and the third in Dresher (1981):

Theorem 1 *(properties of The Riemann Integral).*[5] *Let f and g be continuous functions on interval $\langle a, b\rangle$.*

(i) If $c \in (a, b)$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

(ii) Let $f(x) \geq g(x)$ for every $x \in \langle a, b\rangle$. Then

$$\int_a^b f \geq \int_a^b g.$$

(iii) There exists $\xi \in \langle a, b\rangle$ so that

$$\int_a^b f\,dx = f(\xi)(b - a).^6$$

[5] Author's translation

[6] This property is also known as the 'Mean Value Theorem'.

(iv) Let $c \in \langle a, b \rangle$*. If we denote* $F(x) = \int_c^x f$ *for* $x \in (a, b)$*, then* $F'(x) = f(x)$ *for* $x \in (a, b)$. ♣

Proposition 2 *(properties of The Stieltjes integral). If* f *is continuous and* g' *is Riemann integrable over the specified interval, then*

$$\int f(x)dg(x) = \int f(x)g'(x)dx.$$

♣

Proposition 3 *Let* $\int$ *be a Stieltjes integral and let the Stieltjes integral* $\int_0^1 F(x)dG(x)$ *exist then:*

$$\int_0^1 F(x)dG(x) = E(F(x)).$$

♣

Using the Proposition 3, we receive:

$$(1 + a^i)(1 + r + \alpha_{t+1}) = (1 + \alpha_{t+1})E(1 + a).$$

We know that $E(1+a) = E(1)+E(a) = 1+E(a)$ and recalling that $E(a) = \overline{a}$ we receive:

$$(1 + a^i)(1 + r + \alpha_{t+1}) = (1 + \alpha_{t+1})(1 + \overline{a}),$$

rearranging terms:

$$a^i = \frac{(1 + \alpha_{t+1})}{(1 + r + \alpha_{t+1})}(1 + \overline{a}) - 1. \tag{3.14}$$

Denote such a^i as a^0. For $\forall a^i < a^0$:

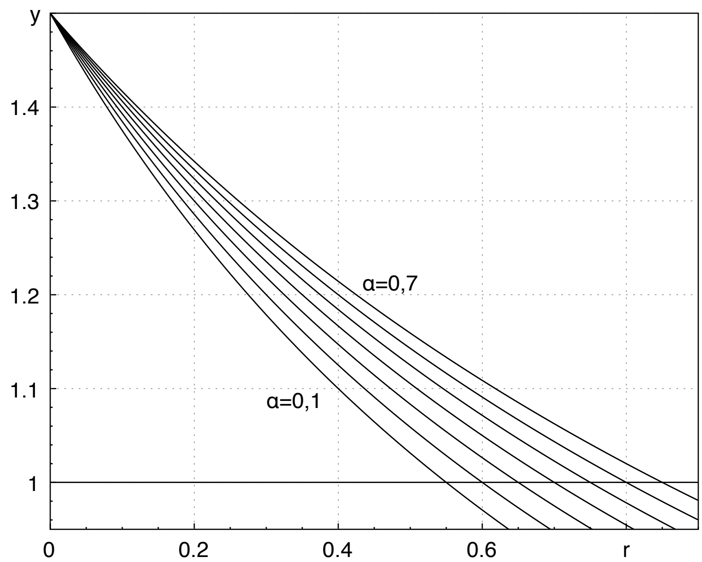
$$(1 + a^i)w\left(1 + \frac{\alpha_{t+1}}{1 + r}\right) < \frac{Lw}{1 + r} \Longrightarrow \frac{\partial w^i}{\partial \tau} > 0.$$

It is trivial for $\forall a^i > a^0$. ♠

Note that for a certain combination of the parameters α and r, the $a^0 \notin \langle 0, 1 \rangle$. We can avoid this problem by assuming r to be reasonably small, yet it would be appropriate to see where these problems arise.

Firstly, we need to know how $\frac{(1+\alpha)}{(1+r+\alpha)}(1+\overline{a})$ behaves, because it gives us the pivotal voter. We see that the fraction is decreasing in r for a fixed α and $1+\overline{a}$, as shown in Figure 3.1. Therefore, if r increases the a^0 decreases, which increases the number of rich young individuals as it becomes more profitable to 'earn' more when young. When r exceeds a certain level relative to a fixed α, there will be no poor young individuals. To be precise there will be only young individuals who support the tax rate τ=0, as they would be better off without any redistribution. These combinations are shown in Figure 3.2 where $\overline{a} = 0.1, 0.3, 0.5, 0.7$ (various distribution of a), and the gray area represents combinations where $a^0 < 0$, for $\overline{a} = 0.5$.

Figure 3.1: $y = \frac{(1+\alpha)}{(1+r+\alpha)}(1+\overline{a})$, $1+\overline{a} = 1.5$

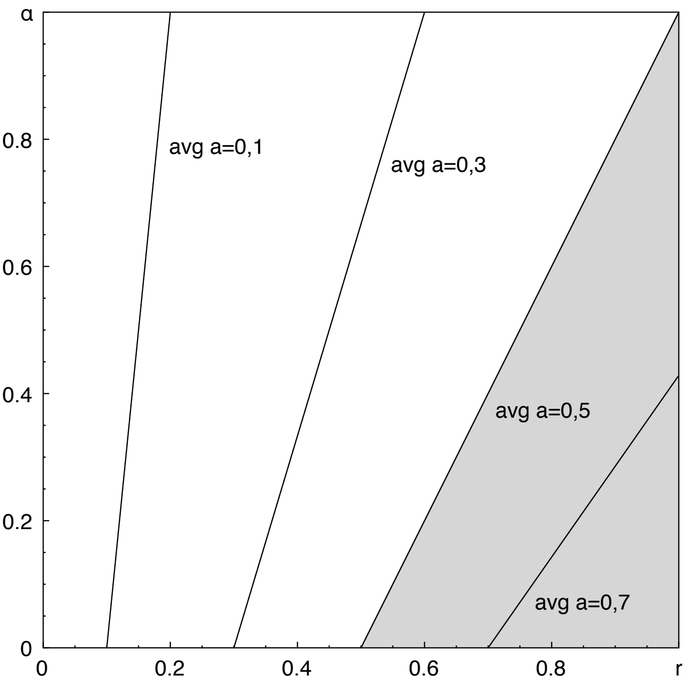


Source: Author's calculations

Example 1 *To keep the notion simple, we assume the parameter a to be uniformly distributed and $r = 0$ for a while. Hence $g(a)$ is a constant function, $g(a) = 1$, for $\forall a \in \langle 0, 1\rangle$:*

$$(1+a^o)(1+r+\alpha_{t+1}) = (1+\alpha_{t+1})\int_0^1 (1+a)dG,$$

Figure 3.2: Combinations of α and r, $a^0 = 0$



Source: Author's calculations

using Proposition 2

$$(1 + a^o)(1 + r + \alpha_{t+1}) = (1 + \alpha_{t+1}) \int_0^1 (1 + a) \cdot 1 da,$$

calculating the integral and rearranging terms:

$$\frac{1 + \alpha_{t+1}}{1 + r + \alpha_{t+1}} = \frac{1 + a^o}{{}^1_0[a + \frac{a^2}{2}]} = \frac{1 + a^o}{1.5}, \quad a^0 = 0.5.$$

As seen, in this setting $a^0 = 0.5$ represents the indifferent individual to changes in τ, in addition a^0 equals a^m and $\overline{a}$. Due to the fact that in the Theorem 1 $f(\xi)$ represents the average value of $f(x)$ on the interval, it holds $\overline{a} = a^0$ because a^i is uniformly distributed (in this example), thus not skewed, it also holds $\overline{a} = a^m$. ♣

Note that the poor young individuals support the tax rate $\tau = 1$ only when the loans against the future benefits are allowed, otherwise they do not support the limiting case of redistribution ($\tau = 1$), as the young individuals (both poor and rich) will not enjoy any consumption when young. This is due to the fact that even though the system redistributes progressively, it does so in an 'intertemporal' manner, thus the redistribution is brought into effect in the next period.

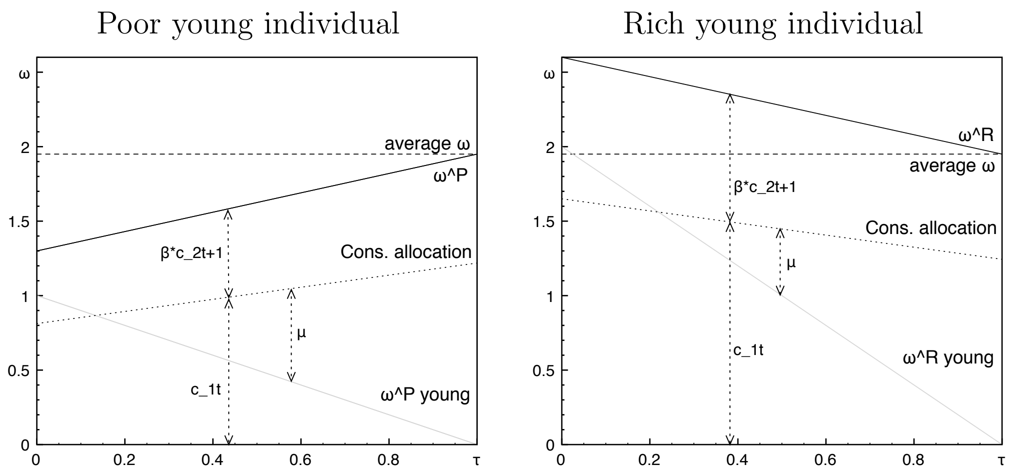
Loans against future benefits To loan a certain amount of money against future benefits, as argued in Mulligan and Sala-i-Martin (1999b), turns out to be very difficult. Perhaps, it is due to the fact that social security is designed to substitute the income when retired and to avoid the reluctance to save enough for retirement when young.

On one hand, if the young individuals are allowed to loan against the future benefits, they will favour the maximal redistribution. On the other hand, we can rationally expect the maximum 'loan' to be a smaller amount than the future benefits. This leads to the situation where the poor young individuals will support the greatest tax rate τ not 'as possible' but 'as they can afford'. So that their first order condition (3.11) holds.

We denote the loan as μ and the maximal loan as μ^m, hence there exists μ^m so that every poor young individual supports a tax rate $\tau^i \in (0,1)$ and his first order condition (3.11) holds. Owing to not knowing the utility function $u(\cdot)$, it is not possible to solve this problem in general.

To explain the idea clearly, in Figure 3.3 there are two graphs of wealth as a function of τ for the individual, with the highest ability paramater a^i and for the individual with the lowest paramater a^i. The actual values are $a^{max} = 2$ and $a^{min} = 1$. In the figure, we assume $r = 0$, $\rho = 0$, and thus from (3.11) follows $c_{1t} = \frac{c_{2t+1}}{\beta_{t+1}}$. As the consumption streams must equal in the optimum, the allocation is depicted by the dot-line. The line marked as 'average wealth' is a wealth of the indifferent individual a^0. Note that in the case of total redistribution $\tau = 1$ the wealth of all individuals in one generation equals and they do not enjoy any consumption when young (if loans are ruled out). The vertical difference between the dot-line and the gray-line represents the required loan μ so that the individual can allocate her consumption (the 'negative' value represents the saving decision).

Figure 3.3: Wealth as a function of τ



Source: Author's calculations

Example 2 *In this example, we illustrate how the poor young individuals behave under such conditions. We assume a following set of parameters and functions:*

$$
\begin{array}{lll}
g(a) = 1 & G(a) = a & \overline{a} = 0.5 \\
u = \log(c) & \alpha_{t+1} = 0.3 & \beta_{t+1} = 0.6 \\
r = 0.03 & \rho = 0.05 & w = 1 \ ,
\end{array}
$$

We calculate how the loan ceiling affects the preferred tax rate by poor young individuals. We concentrate only on the poor young cohort, which is characterised by the equation (3.14) as $a^i < a^0 = 0.4661654$. *Recalling the young individual's first order condition (3.11):*

$$\frac{1}{1+r}u'(c_{1t}) = \frac{1}{1+\rho}Eu'\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big),$$

the consumption constraint in the period t *reads:*

$$c_{1t} = (1-\tau)(1+a^i)w + \mu = (1-\tau)(1+a^i) + \mu,$$

and in the period $t+1$:

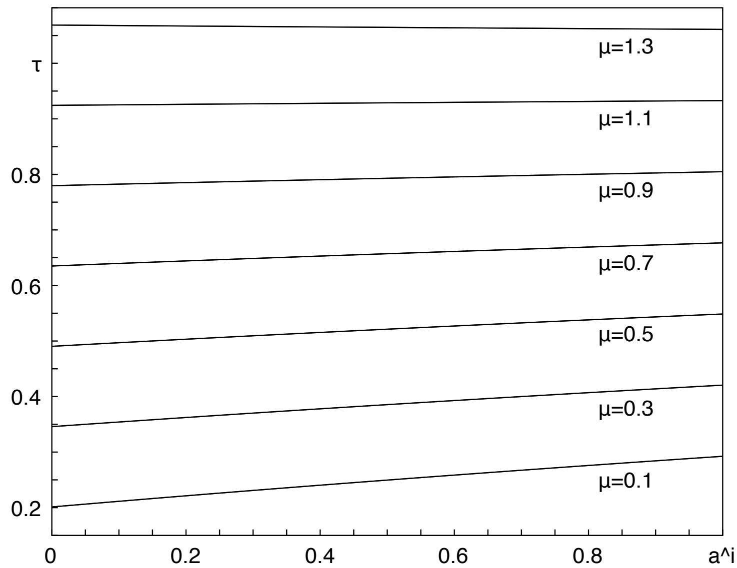
$$c_{2t+1} = \alpha_{t+1}(1-\tau)(1+a^i)+\tau Lw-(1+r)\mu = 0.3(1-\tau)(1+a^i)+1.95\tau-1.03\mu,$$

calculating and stating τ:

$$\tau(a^i,\mu) = \frac{0.505a^i + 2.8325\mu + 0.505}{0.505a^i + 3.9175},$$

we can see the preferred τ as the function of $\tau(a^i,\mu)$. As expected, $\tau(a^i,\mu)$ is increasing in μ which means that the more the poor young individuals can loan, the higher tax rate τ they support. In Figure 3.4, there is a diagram of preferred tax rates τ for fixed $\mu = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3$.

Figure 3.4: Preferred τ for fixed μ



Source: Author's calculations

Note that although τ is neither constant nor linear in a^i, the marginal increase in a^i in $\langle 0, a^0)$ does not change τ very much. ♣

Comparison of optimal α Dividing young individuals into the poor and rich cohort (in their 'young' period) might give us a tool to compare their preferences for α. Since every individuals' budget constraint holds, it follows that the greater wealth, ω_t, the greater $c_{1t} + \frac{c_{2t+1}}{1+r}$. Therefore, the marginal utility of consumption (when young) of a poor young individual is greater than the marginal utility of consumption of a rich young individual. Recalling (3.12), one might expect the rich to prefer an earlier retirement age (smaller parameter α), yet due to the fact that $\omega_t(\cdot)$ is also a function of the ability parameter a^i, the total effect is ambiguous and it is not possible to solve it in general. Hence, we let the government set α in accordance with the utilitarian optimum.

3.2.3 Old Individuals

As old individuals decide on their preferred parameters, they take their past contribution as sunk cost; therefore taking into account only the reduced time horizon. Their direct utility function in time t reads:

$$V_t(\overline{c}_{1t-1}, c_{2t}, \alpha_t, \beta_t) = u(\overline{c}_{1t-1}) + \beta_t\Big(\frac{c_{2t}}{\beta_t}\Big) - \alpha_t v\Big(\frac{\alpha_t}{\beta_t}\Big). \tag{3.15}$$

Note that there is no uncertainty, as the old individuals know the 'life-expectancy' parameter β_t at the beginning of the period t. The old individual's budget wealth is constructed analogously:

$$\omega_t = (1+r)\overline{s}_{t-1} + (1-\tau)w\ell_o^i + b,$$

where $\overline{s}_{t-1}$ represents the savings decision ($s_{t-1} = (1-\tau)w\ell_y^i - c_{1t-1}$) that is determined in time $t-1$ so taken as given in the time t. Hence the old individual's budget constraint is:

$$c_{2t} = (1+r)\overline{s}_{t-1} + (1-\tau)w\ell_o^i + b. \tag{3.16}$$

Maximising[7] the direct utility function of an old individual (3.15) subject to (3.16) we obtain:

$$u'\Big(\frac{c_{2t}}{\beta_t}\Big)\Big[(1-\tau)(1+a^i) + \tau L_y\Big]w = \eta\Big(\frac{\alpha_t}{\beta_t}\Big). \tag{3.17}$$

[7]See Appendix 3 for more details.

The old individual's first order condition to the problem of maximising, varies from the young individual's one, as the old individuals do not have to allocate the consumption as their consumption streams were determined in the previous period. The condition (3.17) gives us the optional allocation between consumption and leisure. Now we turn to the analysis of the preferred tax rate τ. Assuming the parameter α to be preset, we find the preferred tax rate by examining the individual's wealth and comparing the benefits costs. Deriving ω_t we receive:

$$\frac{\partial \omega_t}{\partial \tau} = \frac{\partial}{\partial \tau}\Big[(1+r)\overline{s}_{t-1} + (1-\tau)w\ell_o + b\Big] =$$

$$= \frac{\partial}{\partial \tau}\Big[(1+r)\overline{s}_{t-1} + (1-\tau)\alpha_t(1+a^i)w + \tau L w\Big] =$$

$$= -\alpha_t(1+a^i)w + Lw$$

Similarly to 3.2.2, we try to find a^0 so that $\frac{\partial \omega_t}{\partial \tau} = 0$ for such a^0, yet we see that $\frac{\partial \omega_t}{\partial \tau} > 0$ for $\forall a^i \in \langle 0, 1\rangle$. This has a very interesting interpretation. In this case, all old individuals support the limiting case of redistribution, as the benefits outweigh the costs even for highly able individuals.

Corollary 2 *Let O_t be a set of old individuals in time t. If every $a^i \in O_t$ takes past contributions in time $t-1$ as sunk costs, then*

$$\textit{for } \forall a^i \in O: \quad \frac{\partial \omega_t}{\partial \tau} > 0, \quad \forall \tau \in \langle 0, 1\rangle.$$

♣

Proof We show that

$$\alpha_t(1+a^i)w < Lw, \quad \text{for } \forall a^i \in \langle 0, 1\rangle.$$

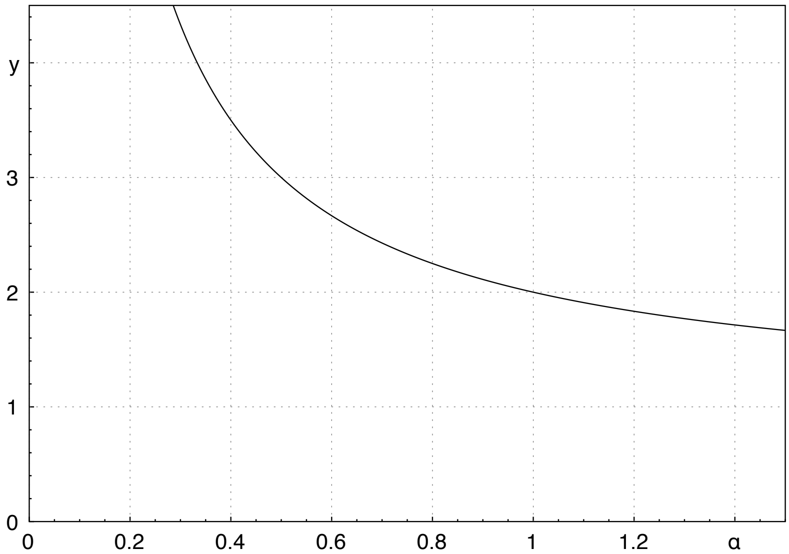
Substituting $(1+\alpha_t)\int_0^1 (1+a)\mathrm{d}G$ for L:

$$\alpha_t(1+a^i) < (1+\alpha_t)\int_0^1 (1+a)\mathrm{d}G,$$

rearranging terms:

$$1 + a^i < \frac{1+\alpha_t}{\alpha_t}\int_0^1 (1+a)\mathrm{d}G,$$

Figure 3.5: $y = \frac{1+\alpha}{\alpha} > 2$, for $\alpha \in (0, 1)$



Source: Author's calculations

we see that $\frac{1+\alpha_t}{\alpha_t}$ is strictly decreasing function. See the Figure 3.5 for the graph. For $\alpha_t \to 1 : \frac{1+\alpha_t}{\alpha_t} \to 2$, therefore:

$$1 + a^i < 2 \int_0^1 (1 + a)\mathrm{d}G.$$

Using the Proposition 3, we receive:

$$1 + a^i < 2E(1 + a),$$

rearranging terms:

$$1 + a^i < 2E(1) + 2E(a),$$

$$1 + a^i < 2 + 2\overline{a}, \tag{3.18}$$

it is clear that for $\forall a^i \in \langle 0, 1)$ the equation (3.18) does hold. For $a^i = 1$ we need to show that $E(a) \neq 0$. If $E(a) = 0$ then we use the Proposition 3 and the Proposition 2:

$$0 = E(a) = \int_0^1 a\mathrm{d}G(a) = \int_0^1 ag(a)\mathrm{d}a = 0.$$

Consequently, we need the following Lemma 4[8] (Johanis (2006)).

Lemma 4 *Let $\varphi \in C\langle 0,1\rangle$ is nonnegative function and $\int_0^1 \varphi(x)dx = 0$ then $\varphi \equiv 0$.* ♣

Proof Denote Φ as the primitive function to φ. We know that:

$$\int_0^1 \varphi(x)\mathrm{d}x = \lim_{x\to 1^-}\Phi(x) - \lim_{x\to 0^+}\Phi(x).$$

Because φ is continuous on $\langle 0,1\rangle$, it holds:

$$0 = \int_0^1 \varphi(x)\mathrm{d}x = \Phi(1) - \Phi(0),$$

$$\Phi(1) = \Phi(0).$$

Differentiating both sides with respect to x:

$$\varphi(1) = \varphi(0).$$

There are two possibilities:

(i) $\varphi(x)$ is a constant function on $\langle 0,1\rangle$, $\varphi(x) = c, \quad c \in \mathbb{R}$.

(ii) $\varphi(x)$ is not a constant[9] function on $\langle 0,1\rangle$, hence there exist n points a_i and $n+1$ open intervals $\mathbb{I}_j$, $n \geq 1$, $n \in \mathbb{N}$, so that

$$a_1..a_n: \quad \varphi'(a_i) = 0 \quad \text{or does not exist,} \quad \bigcup_{i=1}^{n} a_i = \mathbb{A},$$

$$\mathbb{I}_1..\mathbb{I}_{n+1}: \quad \mathbb{I}_j \subset \langle 0,1\rangle, \quad \bigcap_{j=1}^{n+1}\mathbb{I}_j = 0, \quad \mathbb{A} \cup \left(\bigcup_{j=1}^{n+1}\mathbb{I}_j\right) = \langle 0,1\rangle,$$

[8] This is an ordinary lemma for the area of the calculus of variations thus proved, using other lemmas from that area. Therefore, the proof presented in this study is thought to be original and not presented anywhere else.

[9] If there exists an interval within $\langle 0,1\rangle$ where φ is a constant function, we can apply (i) and then apply steps from (ii) on the rest of $\langle 0,1\rangle$.

WLOG we assume two such points a_1 and a_2 so that φ is increasing on $\langle 0, a_1)$ and $(a_2, 1\rangle$ and decreasing on (a_1, a_2). Using Theorem 1 proposition (i):

$$\int_0^1 \varphi(x)\mathrm{d}x = \int_0^{a_1} \varphi(x)\mathrm{d}x + \int_{a_1}^{a_2} \varphi(x)\mathrm{d}x + \int_{a_2}^{1} \varphi(x)\mathrm{d}x = 0,$$

rearranging terms so that intervals over which is φ the increasing function are on the left side and *vice versa*. Recalling that φ is a nonnegative function we use the Theorem 1 property (ii) ($\varphi > 0$ over $\mathbb{I}_j \implies \int_{\mathbb{I}_j} \varphi > \int_{\mathbb{I}_j} 0 = 0$):

$$\underbrace{\int_0^{a_1} \varphi(x)\mathrm{d}x + \int_{a_2}^{1} \varphi(x)\mathrm{d}x}_{>0} = \underbrace{- \underbrace{\int_{a_1}^{a_2} \varphi(x)\mathrm{d}x}_{>0}}_{<0}$$

It is evident that for a nonnegative function it does not hold.

Hence φ is a constant function $\varphi \equiv c$. We calculate c:

$$0 = \int_0^1 c\mathrm{d}x = [cx]_0^1 = c \cdot 1 - c \cdot 0 = c,$$

it is obvious that $c = 0$ therefore $\varphi \equiv 0$. ♣

Applying Lemma 4 on $\int_0^1 ag(a)\mathrm{d}a = 0$ we receive:

$$ag(a) \equiv 0.$$

Since $a > 0$ for $\forall a \in (0, 1\rangle \implies$

(i) $g(a) \equiv 0$ on $(0, 1\rangle$ and $a = 0$; g is a continuous density function $\implies$ $\implies \int_0^1 g(a)\mathrm{d}a = 1 \implies g(0) \gg 0 \implies$ g is not continuous from the right at 0, which is in contradiction with the definition of g.

(ii) $g(a) \equiv 0$ on $\langle 0, 1\rangle \implies \int_0^1 0\mathrm{d}a = 0$ which is in contradiction with the definition of g.

It is evident that $E(a) \neq 0$. We showed that the equation (3.18) holds. ♠

The old individuals therefore prefer the tax rate $\tau = 1$, that is the limiting case of redistribution. This is caused by the assumption of reduced time horizon, that makes even highly able individuals (those 'rich' when young) support that high level of redistribution.

Example 3 *To keep the notion simple, we assume the parameter a to be uniformly distributed and $\alpha = 0.3$ for a while. Hence $g(a)$ is a constant function, $g(a) = 1$, for $\forall a \in \langle 0, 1\rangle$:*

$$\alpha_t(1 + a^i)w < Lw, \quad \textit{for } \forall a^i \in \langle 0, 1\rangle.$$

Substituting $(1 + \alpha_t)\int_0^1 (1 + a)dG$ *for* L*:*

$$\alpha_t(1 + a^i) < (1 + \alpha_t)\int_0^1 (1 + a)dG,$$

Using the Proposition 3 and rearranging terms, we receive:

$$1 + a^i < \frac{1 + \alpha_t}{\alpha} E(1 + a) = \frac{1 + \alpha_t}{\alpha}\Big[E(1) + E(a)\Big]$$

We know that $E(1) = 1$, $E(a) = \overline{a} = 0.5$ and $\alpha = 0.3$:

$$1 + a^i < \frac{13}{3}(1.5) = \frac{13}{2}$$

$$a^i < \frac{13}{2} - 1 = 5.5$$

Hypothetically, the indifferent old individual would be $a^i = 5.5$ and those who would benefit from a decrease in the tax rate, would be $a^i > 5.5$. As we suppose individuals only with the ability parameter $a^i \in \langle 0, 1\rangle$, we see that the whole old cohort prefers the tax rate $\tau = 1$. ♣

3.2.4 Increases in Longevity

Before we proceed to the analysis of the effects of changes in longevity, we will have a closer look at the government's optimum. The first order condition of optimum (3.8):

$$u'\Big(\frac{c_{2t}}{\beta_t}\Big) = \frac{1+r}{1+\theta} Eu'\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big),$$

implies that for $\theta > r$ the marginal utility of consumption for the future old generation in time $t+1$, will be higher than for the old generation in time t. From the equation (3.7) and (3.8) follows:

$$u'(c_{1t}) = u'\Big(\frac{c_{2t}}{\beta_t}\Big) = \frac{1+r}{1+\theta} Eu'\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big).$$

That means that the young generation have smaller consumption than the old. The adjustments in α are based on $u'(\frac{c_{2t}}{\beta_t})wL_y = \eta(\frac{\alpha_t}{\beta_t})$, and because we know that $u'(\frac{c_{2t}}{\beta_t}) < Eu'(\frac{c_{2t+1}}{\beta_{t+1}})$ and the equation (3.10), it follows that:

$$\eta\Big(\frac{\alpha_t}{\beta_t}\Big) < E\eta(\frac{\alpha_{t+1}}{\beta_{t+1}}),$$

and thus $\alpha_t < \alpha_{t+1}$, *ceteris paribus.*

Longevity The changes in longevity affect both the retirement age α and the consumption stream when old. We focus on increases in longevity as it is empirically more relevant. The increase in longevity is in our model represented as $\beta_t < \beta_{t+1}$. The government automatically reflects the change through the first order condition (3.10) that implies an increase in α.

Andersen (2006) suggests that it is not feasible to maintain these objectives: unchanged consumption c_{1t+j} for young individuals, unchanged consumption stream of old individuals $\frac{c_{2t+j}}{\beta_{t+j}}$ and the relative retirement allocation $\frac{\alpha_{t+j}}{\beta_{t+j}}$. To show this, we recall the aggregate resource constraint (3.6) for one period that is:

$$wL(\alpha_{t+j}) = c_{1t+j} + c_{2t+j},$$

rearranging terms and applying $L(\alpha_{t+j}) = (1+\alpha_{t+j})(1+\overline{a})$, which has been shown above, we receive:

$$(1+\overline{a})w + \alpha_{t+j}(1+\overline{a})w = c_{1t+j} + c_{2t+j},$$

$$c_{1t+j} = (1+\overline{a})w - \beta_{t+j}\left[\frac{c_{2t+j}}{\beta_{t+j}} - \frac{\alpha_{t+j}(1+\overline{a})w}{\beta_{t+j}}\right].$$

This means that the consumption of young is equal to their income minus the longevity of the old, times the difference between the consumption and income for old. We see that $\left[\frac{c_{2t+j}}{\beta_{t+j}} - \frac{\alpha_{t+j}(1+\overline{a})w}{\beta_{t+j}}\right] > 0$, therefore it implies transfers to the old individuals. The objective to maintain c_{1t+j}, $\frac{c_{2t+j}}{\beta_{t+1}}$ and $\frac{\alpha_{t+j}}{\beta_{t+j}}$ unchanged is not possible. Note that this implication does not rely on any specification of the model, but follows directly from basic budget mechanism. Andersen (2006) concludes that the problem of how to adjust the social security system to increasing longevity cannot simply be solved by indexing key parameters of the social security system to longevity.

3.2.5 Concluding Remarks

Mapping the individuals preferences that are determined by the way the model was built, we find common implications. Predominantly, these implications are represented by the preferences of the rich and the poor young individuals who support no redistribution and total redistribution, respectively. Employing a loan ceiling (which is supported by Mulligan and Sala-i-Martin (1999b)) allows us to aggregate the individuals' preferences through a political process. Moreover, we receive the commonly observed fact that old individuals support high level of redistribution.

Size of Cohorts Generally speaking, we can quite easily estimate the ratio of poor and rich young individuals to the whole population in time t, as we know the indifferent individual a^0:

$$P^Y \text{ to the population} : \gamma^P = \frac{G(a^0)}{1+\beta}, \tag{3.19}$$

$$R^Y \text{ to the population} : \gamma^R = \frac{1-G(a^0)}{1+\beta}. \tag{3.20}$$

The size of the old cohort is unity, yet as they do not live throughout the whole period, their share[10] is thus:

$$O \text{ to the population} : \gamma^O = \frac{\beta}{1+\beta}. \tag{3.21}$$

[10]This notion can be also explain as the weighted vote.

Correspondingly, the ratios are denoted as γ^J, $J = \{R, P, O\}$ For the whole population in a period holds: $\sum_J \gamma^J = 1 + \beta$. We denote the preference of cohorts for the tax rate τ as:

- Rich young: $\tau_R = 0$,
- Poor young: $\tau_P = 1$ or if μ is restricted $\tau_P \in \langle 0, 1 \rangle$,
- Old: $\tau_O = 1$.

3.3 Electorate Competition

In this section, we focus on aggregation of cohorts' preferences. Due to the outcome that our model gives, we cannot use the median voter concept. The probabilistic voting will serve our purposes quite well. Despite having two groups supporting one policy each, the poor young cohort's preferences may not be the same for every individual amongst the cohort. We can solve this drawback by restricting the maximum loan μ and binding the poor young cohort's preferred policy. Therefore, we obtain the third group supporting one policy. Considering this adjustment, we apply the probabilistic voting.

3.3.1 Preferred Policy

The situation of the rich young cohort and the old cohort is very simple. The ambiguity comes up in the poor young cohort. As mentioned above, the poor young individuals prefer either $\tau = 1$ in the case where loans against the future benefits are allowed or $\tau_P \in \langle 0, 1 \rangle$ in the case where there is a maximal loan μ. Therefore, we assume three cases: the first $\tau_P = 0$; the second we assume μ so that average preferred tax rate τ amongst the poor young individuals is $\tau_P = 0.5$; and the third $\tau_P = 1$. There is a recapitulation in Table 3.1:

Knowing the relative sizes of cohorts, γ^J, we proceed to the voting model. The following subsection builds on Persson and Tabellini (2000), pg. 52–58.

Table 3.1: Preferred τ in Different Scenarios

	Scenario 1	Scenario 2	Scenario 3
τ_R	0	0	0
τ_P	0	0.5	1
τ_O	1	1	1

Source: Author's calculations

3.3.2 Probabilistic Voting

As mentioned in Section 2.2, the probabilistic voting concept adds the extra dimension of 'ideology' to the voting. Voters across cohorts perceive this dimension differently and the dimension could be interpreted as, for example, personal characteristics of a party leader, bias towards a party or simply candidates' popularity.

We proceed as Persson and Tabellini (2000), we have a population consisting of three groups, $J = \{R, P, O\}$, which are the rich young individuals, the poor young individuals and the old individuals, respectively. Every group supports a policy τ_J, and has got a population share γ^J. Furthermore, we need to define a function that represents a group's preference over a policy. Such function is concave and has maximum on $\langle 0, 1\rangle$ in τ_J. We denote it as $W^J(\tau) \equiv V(c_{1t}(\tau), c_{2t+1}(\tau))$ for $J = \{R, P\}$, and $W^O(\tau) \equiv V(\bar{c}_{1t-1}(\tau), c_{2t}(\tau))$ for the old individuals. There are two candidates competing in the election party A and party B, and proposing policies τ_A and τ_B, respectively.

At the time of the elections, voters decide considering both the economic policy announced and the candidates' ideologies. Consequently, the voter i in group J prefers the candidate A if:

$$W^J(\tau_A) > W^J(\tau_B) + \sigma^{iJ} + \delta, \tag{3.22}$$

where σ^{iJ} is each individual's specific parameter that can take on negative as well as positive values. It measures the individual's bias towards the candidate B, the positive value means that the individual has a bias favouring the party B, whereas $\sigma^{iJ} = 0$ indicates that the individual is indifferent to ideology and cares only about policy. We assume that this parameter has group-specific uniform distribution on:

$$\Big\langle -\frac{1}{2\phi^J}, \frac{1}{2\phi^J} \Big\rangle.$$

These distributions thus have density ϕ^J, which means that each group has members biased towards both candidates. The parameter δ takes on negative and positive values as well. It measures the average 'popularity' of the candidate B amongst all individuals. We also assume a uniform distribution on:

$$\Big\langle -\frac{1}{2\chi}, \frac{1}{2\chi} \Big\rangle.$$

The timing of events is as follows:

1. The two candidates, simultaneously and non-cooperatively, announce their policies: τ_A, τ_B. At this stage they are aware of voters' preferences and even of distributions of σ^{iJ} and δ, but not yet the actual values.

2. The actual values are realised and there is no longer any uncertainty.

3. The elections are held.

4. The elected candidate implements his announced policy.

In the framework of probabilistic voting, it is crucial to identify the 'swing' voter in a group J. Given the candidates' policies, the swing voter has such ideological bias that it makes him indifferent to the two parties:

$$\sigma_J = W^J(\tau_A) - W^J(\tau_B) - \delta. \tag{3.23}$$

Consequently, all voters in group J with $\sigma^{iJ} \leq \sigma^J$ prefer the party A. Applying our assumptions about distributions, the candidate A's vote share is:

$$\pi_A = \sum_J \gamma^J \phi^J \Big(\sigma^J + \frac{1}{2\phi^J}\Big) \tag{3.24}$$

Since the realised value of δ affects σ^J (σ^J depends on δ), the vote share of a candidate is also a random variable. Therefore, both candidates perceive the elections as a random event, related to the realisation of δ.[11] Correspondingly, the probability of the candidate A is:

$$p_A = \text{Prob}_\delta\left[\pi_A \geq \frac{1}{2}\right] = \frac{1}{2} + \frac{\chi}{\phi}\left[\sum_J \gamma^J \phi^J \left[W^J(\tau_A) - W^J(\tau_B)\right]\right], \qquad (3.25)$$

where $\phi = \sum_J \gamma^J \phi^J$ is the average density across groups. Evidently, the candidate B wins with probability $1 - p_A$. This notion demonstrates a great advantage of probabilistic voting. As both individuals' utility and distribution of ideology preferences are continuous functions, the probability of winning becomes a smooth function of the distance between the two electoral platforms (Persson and Tabellini (2000)).

The unique equilibrium is characterised by the convergence of both candidates to the same policy, which is caused by the fact that both candidates face the same optimalisation problem, only with opposite signs. Needless to say, both candidates compete for an exogenous rent that does not enter our model. Moreover both candidates have the same concave preferences and the same 'technology' for turning their resources into votes.

In equilibrium, both candidates maximise the weighted social welfare function, the weights are γ^J and ϕ^J. γ^J represents the share of the cohort and ϕ^J represents the group specific density (with mean equal to zero), i.e. how voters are responsible in a group J to the announced policies and how they reward it with votes at the elections. In Figure 3.6, there are depicted the distribution of σ^{iJ} amongst groups. The height of the distribution symbolises the density ϕ^J. Persson and Tabellini (2000) suggest that it represents how many voters are gained in the group per marginal increase in $W(\tau)$.

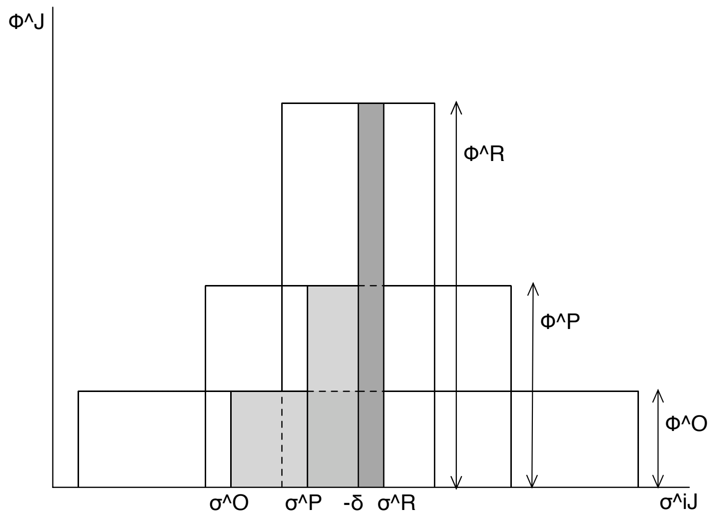
We also assume $\phi^O < \phi^P < \phi^R$. The old individuals do no have that much 'in stake' and from their perspective the decision on τ affects only part of their live; whereas the young individuals take the decision on τ as

[11] Persson and Tabellini (2000) suggest the distribution of δ not to be so narrow, so that corner solutions are ruled out.

granted also for the next period. The difference between the poor young and the rich young cohort arises from the possible harm, that the changes in τ may bring about; the rich young are net contributers, and thus they keep an eye on the proposed τ most.

The equilibrium swing voter is $\sigma^{iJ} = -\delta$, the voters σ^{iJ} to the left of $-\delta$ support the party A and the rest the party B.

Figure 3.6: Densities ϕ^J per group

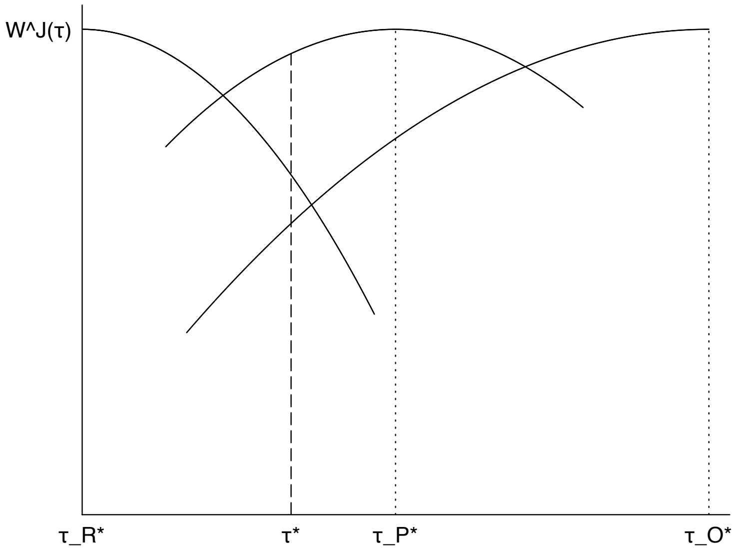


Source: Persson and Tabellini (2000), pg. 55.

As in Persson and Tabellini (2000), we now consider a small deviation from the equilibrium policy by the candidate A, announcing a lower τ. The swing voter becomes, as depicted in Figure 3.6, an individual with σ^R, σ^P and σ^O in rich, poor and old cohort, respectively. The utility is illustrated in Figure 3.7, where τ^* is the equilibrium policy.

A relative small change in the announced policy by the candidate A, brings a relative large change in vote share in the rich group. This is caused by high density ϕ^R, making the rich individuals caring more about policy, and thus more vulnerable to 'office seeking' candidates. Correspondingly, the lower densities of poor and old groups, make these groups less volatile to

Figure 3.7: Voters' Utility



Source: Adjusted Figure from Persson and Tabellini (2000), pg. 56.

changes in the announced policy. The outcome of the application of probabilistic voting on our model of redistribution is presented in the following subsection.

3.3.3 Application and Interpretation

In the context of our model, we see that we have got two groups[12] supporting tax rate greater than zero. Not considering its value, we can say that a redistributive social security system would be enacted and then sustained over time.[13]

Applying the probabilistic voting concept, we need to assume that all young individuals believe that the tax rate (the elections' outcome) will be sustained in the next period. Otherwise, both young groups will not support

[12]The group of poor young and of old individuals, these two groups can form a majority under assumption of lower r.

[13]We assume r to be rationally small, as explained above.

any redistribution as it will be very likely that the future young individuals will not support it. If we substitute this assumption by the assumption that the old generation is not affected[14] by the elections, we will face common vote of young generation on the rate of redistribution. Besides, there would probably exist two different tax rates: one set in elections in time t and the other in $t-1$; and these two rates would be valid only for a certain group each. Note that this notion cannot be simulated by our model.

Therefore, we assume the young individuals to believe that in the next period there will not be any changes to τ (or at least not any drastic ones), so that they vote as if τ remains the same. Generally speaking, the outcome of the probabilistic voting suggests that a group taking announced policy more seriously is an easy prey for office seeking candidates. Such group has got more swing voters, those willing to sway their vote for a little change in policy. In votes on redistribution, these groups are usually formed by high income individuals and characterised by high densities ϕ (Persson and Tabellini (2000)).

On one hand, the rich young individuals have the highest density ϕ as they have the most 'in stake', on the other hand the old individuals have the lowest density as the marginal increase in their utility caused by the increase in τ is the lowest, because it affects only the consumption when old.[15] The outcomes differ in our three scenarios specified in Table 3.1.

Scenario 1 The first scenario assumes the poor young individuals to prefer $\tau_P = 0$, which is not very likely. The poor young individuals form a coalition with the rich young individuals. Hence we receive two groups of generally different sizes, preferring opposite level of redistribution; young cohort and old cohort. Their densities are ϕ^Y and ϕ^O; knowing that it holds $\phi^R > \phi^P > \phi^Y \implies \phi^Y > \phi^O$. The sizes of cohorts are simply $\gamma^Y = \frac{1}{1+\beta}$ and $\gamma^O = \frac{\beta}{1+\beta}$, for the young and old cohort, respectively. In the sense of the probabilistic

[14]They have to pay the tax τ set in time $t-1$ and their benefits are also determined by $t-1$.

[15]$V(\cdot)$ is concave $: \Delta\omega \implies V(c_{1t} + \frac{\Delta}{2}, c_{2t+1} + \frac{\Delta}{2}) \geq V(c_{1t}, c_{2t+1} + \Delta)$

voting, the equilibrium policy favours the young voters, i.e. τ is close to zero. Despite the young individuals' winning the elections, the outcome τ is not equal to zero, which means that even under these circumstances the social security system would be sustained.

Scenario 2 This setting reflects more realistic situation and that is when the poor young individuals favour positive tax rate, but their willingness to support $\tau = 1$ is bounded by the restriction on the maximal loan μ.[16] The relative sizes of the groups are determined by the distribution of the ability parameter a, by the life expectancy β and by r.

Example 4 *We assume the parameter a to be uniformly distributed,* $\beta = 0.6$ *and* $r = 0$. *We receive* $a^0 = 0.5$. *Hence* $\gamma^R = \frac{1-G(a^0)}{1+\beta} = 0.3125$, $\gamma^P = \frac{G(a^0)}{1+\beta} = 0.3125$ *and* $\gamma^O = \frac{\beta}{1+\beta} = 0.375$. *The relative size appears balanced, thus the equilibrium policy is close to the* 0.5, *whether it is closer to* τ_O *or to* τ_R *depends on the actual value of* $\gamma^J \phi^J$ *and finally on the realisation of* δ. ♣

Importantly, we again receive the social security that would be sustained.

Scenario 3 In this scenario, the poor young individuals support the total redistribution as there is no restriction on the maximal loan. Similarly as in Scenarion 1, the poor young individuals form a coalition, yet now with the old individuals. The densities are ϕ^R and ϕ^{P+O}. From $\phi^R > \phi^P > \phi^Y$ follows $\phi^R > \phi^{P+O}$. The relative sizes are $\gamma^R = \frac{1-G(a^0)}{1+\beta}$ and $\gamma^{P+O} = \frac{G(a^0)+\beta}{1+\beta}$. Note that increasing r and decreasing β cause increases in γ^R. Hence there may exist a threshold combination of r and β, so that the rich young individuals with higher density may be the group whose high value of $\gamma^R \phi^R$ attracts candidates so that the equilibrium policy is closer to τ_R. Importantly, even in this situation it does not mean zero tax rate, and thus no redistribution. As a result, the social security would be sustained.

[16] We assume all individuals in this group to prefer the same τ, yet with the restriction on μ they would slightly vary in preferred τ by each individual. We neglect these differences to simplify the usage of the probabilistic voting model.

Longevity and Vote Shares The changes in longevity affect the election by shuffling the relative sizes of groups. We again discuss only the increases in β. As β increases, the relative size of the old cohort rises up to $\frac{1}{2}$; $\lim_{\beta\to 1}\frac{\beta}{1+\beta}=\frac{1}{2}$, whereas the relative sizes of rich and poor young individuals diminishes down to $\frac{\gamma^R+\gamma^P}{1+\beta}=\frac{1}{2}$; $\lim_{\beta\to 1}\frac{\gamma^R+\gamma^P}{1+\beta}=\frac{1}{2}$. Therefore, we can expect the equilibrium policy to favour more and more the old young individuals as β rises.

The increasing longevity strengthens the political forces that make the redistribution positive and the social security be sustained. Furthermore, the old individuals get larger influence on the elections' outcome, and thus the tax rate will rise, which will make the social security size grow.

Chapter 4

Conclusion

In this paper we have reviewed literature dealing with various positive models of social security, that aspire to give an explanation why such intertemporal redistributive system originate and then sustain. We discussed various politico-economic settings so that we finally come to a few concepts that we used in the Model of Social Security, presented in Chapter 3.

The main contribution of the paper is the model that combines the standard OLG model, the demographic mechanisms and the probabilistic voting. This approach is, to the author's knowledge, unique and its flexibility allows us to capture various empirically observed effects.

The model connected through heterogeneous population, concepts of reduced time horizon and within cohort redistribution so that finally we could apply the concept of probabilistic voting to aggregate preferences of the individuals. Moreover, it tested the influences of the changes in longevity, that appears to be empirically relevant.

In conclusion, we found, not so surprisingly, that those young individuals with high incomes (those with ability parameter greater than a certain level) favour no redistribution, whereas the other young individuals do favour total redistribution. Consequently, when we employed the cap on loan against future benefits, the poor young individuals lowered their demands and started to prefer a lower level of redistribution. In accordance with our expectations the old individuals support total redistribution, due to the reduced time horizon concept.

Having mapped the individuals' preferences, we turned to their aggregation. The probabilistic voting concept served our purposes well and we discussed three scenarios. In Section 3.3.3, to prevent the conclusions from depending directly on characteristics of the model, we examined the outcomes under three scenarios representing three possible situations. The scenarios differed in the preferred tax rate by the poor young individuals; these differences are caused by the restriction on the loan ceiling.

Nevertheless, we concluded that the social security system (a nonnegative rate of redistribution) would be sustained in all scenarios. This model answers the question from Chapter 1; what politico-economic forces are behind such a massive redistribution. It is the within cohort redistribution that reinforces the power to either leave the system unchanged or to give rise to a new one.

Importantly, the model is consistent with many 'empirical facts' presented in Mulligan and Sala-i-Martin (1999b). To be precise, it accords with: 'social security is financed with special payroll tax', 'social security is mostly PAYG and redistributive across cohorts' and 'it is difficult to borrow against future benefits'.

In addition, the paper contributes to verify the effects of increases in longevity on the social security size. Through the mechanism of the model, the increase in longevity increases the vote share of the old individuals, thus they affect the elections to their favour and therefore the social security size grows.

Lastly, this paper opened a wide space for research of other empirically observed effects in the framework of the model. The environment can analyse another demographic variables, such as population growth and aging society. These variables can be examined in further research.

Appendix A

Mathematical Background

Derivation 1 (Government's Optimum) *To solve the given problem of maximising, we use the Lagrange multiplier method (Kalenda (2005)) and (Hájková, John and Zelený (2000)). We need to maximise the following function:*

$$\Psi_t(\mathbf{x}_t) = (1+\theta)E_t\Bigg[u(\bar{c}_{1t-1}) + \sum_{j=0}^{\infty}\frac{S(\mathbf{x}_{t+j})}{(1+\theta)^j}\Bigg],$$

$$\textit{subject to } \sum_{j=0}^{\infty}\Big(wL(\alpha_{t+j}) - c_{1t+j} - c_{2t+j}\Big)(1+r)^{-j} = 0.$$

We are not interested in ∞ number of solution nor periods, so we choose m sufficiently large so that we can neglect all discounted values of $\frac{S(\mathbf{x}_{t+j})}{(1+\theta)^j}$ for $j > m$.[1] *Hence we write the new problem of maximasing:*

$$\Psi_t(\mathbf{x}_t) = (1+\theta)E_t\Bigg[u(\bar{c}_{1t-1}) + \sum_{j=0}^{m}\frac{S(\mathbf{x}_{t+j})}{(1+\theta)^j}\Bigg],$$

$$\textit{subject to } \sum_{j=0}^{m}\Big(wL(\alpha_{t+j}) - c_{1t+j} - c_{2t+j}\Big)(1+r)^{-j} = 0.$$

The Lagrange function is:

$$L^g(\mathbf{x}_t, \lambda) = (1+\theta)E_t\Bigg[u(\bar{c}_{1t-1}) + \sum_{j=0}^{m}\frac{S(\mathbf{x}_{t+j})}{(1+\theta)^j}\Bigg] -$$

[1] We have got finite number of parameters.

$$-\lambda \sum_{j=0}^{m} \left[\Big(wL(\alpha_{t+j}) - c_{1t+j} - c_{2t+j} \Big) (1+r)^{-j} \right],$$

where λ is the Lagrange multiplier. We need to denote the Lagrange function differently so that it does not coincide with L as the labour supply, hence we denote it as L^g. The first order conditions of optimum are:

The living individuals

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial c_{1t}} = u'(c_{1t}) - \lambda = 0,$$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial c_{2t}} = u'(c_{2t}) - \lambda = 0,$$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial \alpha_t} = -\eta\Big(\frac{\alpha_t}{\beta_t}\Big) + \lambda w \frac{\partial L(\alpha_t)}{\alpha_t} = -\eta\Big(\frac{\alpha_t}{\beta_t}\Big) + \lambda w L_y = 0.$$

The next period

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial c_{1t+1}} = \frac{Eu'(c_{1t+1})}{1+\theta} - \frac{\lambda}{1+r} = 0,$$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial c_{2t+1}} = \frac{Eu'(c_{2t+1})}{1+\theta} - \frac{\lambda}{1+r} = 0,$$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial \alpha_{t+1}} = -\frac{E\eta(\frac{\alpha_{t+1}}{\beta_{t+1}})}{1+\theta} + \lambda w L_y = 0.$$

The future individuals, for $m > j > 2$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial c_{1t+j}} = \frac{Eu'(c_{1t+j})}{(1+\theta)^j} - \frac{\lambda}{(1+r)^{-j}} = 0,$$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial c_{2t+j}} = \frac{Eu'(c_{2t+j})}{(1+\theta)^j} - \frac{\lambda}{(1+r)^{-j}} = 0,$$

$$\frac{\partial L^g(\mathbf{x}_t,\lambda)}{\partial \alpha_{t+j}} = -\frac{E\eta(\frac{\alpha_{t+j}}{\beta_{t+j}})}{(1+\theta)^j} + \lambda w L_y = 0.$$

These conditions can be easily rearranged into the form presented in Chapter 3.

Derivation 2 (Young Individuals' Optimum) *We proceed as in Derivation 1 and use the Lagrange multiplier method. The young individuals' direct utility function to be maximised is:*

$$V_t(c_{1t}, c_{2t+1}, \alpha_{t+1}) = u(c_{1t}) + \frac{1}{1+\rho} E_t \Big[\beta_{t+1} u\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big) - \alpha_{t+1} v\Big(\frac{\alpha_{t+1}}{\beta_{t+1}}\Big) \Big],$$

$$\textit{subject to } c_{1t} + \frac{c_{2t+1}}{1+r} - (1-\tau)(1+a^i)w\Big(1 + \frac{\alpha}{1+r}\Big) - \frac{b}{1+r} = 0.$$

The Lagrange function is:

$$L^g(c_{1t}, c_{2t+1}, \alpha_{t+1}, \lambda) = u(c_{1t}) + \frac{1}{1+\rho} E_t \Big[\beta_{t+1} u\Big(\frac{c_{2t+1}}{\beta_{t+1}}\Big) - \alpha_{t+1} v\Big(\frac{\alpha_{t+1}}{\beta_{t+1}}\Big) \Big] -$$

$$-\lambda \Big[c_{1t} + \frac{c_{2t+1}}{1+r} - (1-\tau)(1+a^i)w\Big(1 + \frac{\alpha}{1+r}\Big) - \frac{b}{1+r} \Big],$$

where λ is the Lagrange multiplier. The first order conditions are:

$$\frac{\partial L^g(c_{1t}, c_{2t+1}, \alpha_{t+1}, \lambda)}{\partial c_{1t}} = u'(c_{1t}) - \lambda = 0,$$

$$\frac{\partial L^g(c_{1t}, c_{2t+1}, \alpha_{t+1}, \lambda)}{\partial c_{2t+1}} = \frac{Eu'(c_{2t+1})}{1+\rho} - \frac{\lambda}{1+r} = 0,$$

$$\frac{\partial L^g(c_{1t}, c_{2t+1}, \alpha_{t+1}, \lambda)}{\partial \alpha_{t+1}} = -\frac{E\eta(\frac{\alpha_{t+1}}{\beta_{t+1}})}{1+\rho} + \frac{\lambda}{1+r} \Big[(1-\tau)(1+a^i) + \tau L_y \Big] w = 0.$$

These conditions can be easily rearranged into the form presented in Chapter 3.

Derivation 3 (Old Individuals' Optimum) *One last time, we use the Lagrange multiplier method. The old individuals' direct utility function to be maximised is:*

$$V_t(\bar{c}_{1t-1}, c_{2t}, \alpha_t) = u(\bar{c}_{1t-1}) + \beta_t u\Big(\frac{c_{2t}}{\beta_t}\Big) - \alpha_t v\Big(\frac{\alpha_t}{\beta_t}\Big),$$

$$subject\ to\ c_{2t} - \overline{s}_{1t-1}(1+r) - \alpha(1-\tau)(1+a^i)w - b = 0.$$

The Lagrange function is:

$$L^g(\overline{c}_{1t-1}, c_{2t}, \alpha_t, \lambda) = u(\overline{c}_{1t-1}) + \beta_t u\Big(\frac{c_{2t}}{\beta_t}\Big) - \alpha_t v\Big(\frac{\alpha_t}{\beta_t}\Big)$$

$$-\lambda\Big[c_{2t} - \overline{s}_{1t-1}(1+r) - \alpha(1-\tau)(1+a^i)w - b\Big],$$

where λ *is the Lagrange multiplier. The first order conditions are:*

$$\frac{\partial L^g(\overline{c}_{1t-1}, c_{2t}, \alpha_t, \lambda)}{\partial c_{2t}} = u'(c_{2t}) - \lambda = 0,$$

$$\frac{\partial L^g(\overline{c}_{1t-1}, c_{2t}, \alpha_t, \lambda)}{\partial \alpha_t} = -\eta\Big(\frac{\alpha_t}{\beta_t}\Big) + \lambda\Big[(1-\tau)(1+a^i) + \tau L_y\Big]w = 0.$$

These conditions can be easily rearranged into the form presented in Chapter 3.

Bibliography

Andersen, T. M.: *Increasing Longevity and Social Security Reforms*, CESifo Working Paper no. 1789, August 2006

Andersen, T. M.: *Social Security and Longevity*, CESifo Working Paper no. 1577, October 2005

Azariadis, C., Galosso, C.: *Fiscal Constitutions*, Journal of Economic Theory 103, 255–281, January 2002

Becker, G. S.: *Public Policies, Pressure Groups and Deadweight costs*, Journal of Public Economics 28, 329–347, 1985

Bergstrom, T. C., Hartman, J. L.: *Demographics and the Political Sustainability of Pay-as-you-go Social Security*, CESifo Working Paper no. 1378, January 2005

Bohn, H.: *Social Security and Demographic Uncertainty: The Risk Sharing Properties of Alternative Policies, in Risk Aspects of Investment Based Social Security Reform*, National Bureau of Economic Research Working Paper no. 7030, March 1999

Borck, R.: *Voting, Inequality, and Redistribution*, DIW Working Paper, DIW Berlin, October (2005)

Corneo, G., Gruner, H. P.: *Social Limits to Redistribution*, The American Economic Review, Volume 90, no. 5, December 2000

De Donder, Ph., Hindriks, J.: *Equilibrium Social Insurance with Policy-Motivated Parties*, Département des Sciences Économiques de l'Université catholique de Louvain, Discussion Paper 2006-18, February 2006

Dresher, M.: *The Mathematics of Games of Strategy: Theory and Applications.*, Research Matematician The Rand Corporation, Dover Publications, Inc., New York, 1981

Galosso, V., Profeta, P.: *The Political Economy of Social Security: a Survey*, European Journal of Political Economy, Volume 18, 1 – 29, May 2002

Galosso, V., Profeta, P.: *Lessons for an Aging Society: the Political Sustainability of Social Security Systems*, IGIER – Università Bocconi, Working Paper no. 244, October 2003

Gonzales-Eiras, M., Niepelt, D.: *Sustaining Social Security*, CESifo Working Paper no. 1494, July 2005

Hansson, I., Stuart, C.: *Social Security as Trade among Living Generations*, The American Economic Review, Volume 79, no. 5., 1182–1195, December 1989

Hájková, V., John, O., Zelený, M.: *Matematika*, Univerzita Karlova v Praze, Nakladatelství Karolinum, Praha, 2000

Johanis, M.: *Lecture Handouts – Mathematics IV*, Institute of Economic Studies, Charles University, December 2006

John, O., Kalenda, F. K, Zelený, M.: *Matematika (pokračování)*, MATFYZPRESS, vydavatelství Matematicko-fyzikální fakulty Univerzity Karlovy v Praze, 2003

Kalenda, F. K.: *Materiály k Přednáškám Kurzu Matematika II*, Institute of Economic Studies, Charles University, March 2005

Mulligan, C. B., Sala-i-Martin, X.: *Gerontocracy, Retirement and Social Security*, National Bureau of Economic Research Working Paper no. 7117, May 1999a

Mulligan, C. B., Sala-i-Martin, X.: *Social Security in Theory and Practice (I): Facts and Political Theories*, National Bureau of Economic Research Working Paper no. 7118, May 1999b

Mulligan, C. B., Sala-i-Martin, X.: *Social Security in Theory and Practice (II): Efficiency Therories, Narrative Theories and Implacations for reform*, National Bureau of Economic Research Working Paper no. 7119, May 1999c

Persson, T., Tabellini, G.: *Political Economics: Explaining Economic Policy*, MIT Press, Cambridge, MA., 2000

Profeta, P.: *Demography, Retirement and the Future of Social Security*, Econpubblica, Università Bocconi, working Paper no. 68, March 2000

Profeta, P.: *Aging and Retirement: Evidence Across Countries*, Università di Pavia and Università Bocconi, Working Papers no. 123/2002, March 2002

Razin, A., Sadka, E.: *Aging and the welfare state: The Role of the Young and the Old Voting Pivots*, National Bureau of Economic Research Working Paper no. 10967, December 2004

Sjoblom, K.: *Voting for Social Security*, Public Choice 45, 225 – 240, 1985

Sinn, H., Uebelmesser, S.: *Pensions and the Path to Gerontocracy in Germany*, European Journal of Political Economy, Volume 19, 153 – 158, May 2002

Tabellini, G.: *A Positive Theory of Social Security*, The Scandinavian Journal of Economics 102, 523 – 545, 2000

Uebelmesser, S.: *Political Feasibility of Pension Reforms*, Center for Economics Studies, University of Munich, and CESifo, December 2003

Weisstein, E. W.: *Stieltjes Integral.*, MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/StieltjesIntegral.html, April 2006