POWER, FREEDOM, AND VOTING

CONCEPTUAL, FORMAL, AND APPLIED DIMENSIONS

Edited by

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2. Power Indices Methodology: Decisiveness, Pivots, and Swings

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1. Introduction

Three most frequently used measures of a priori voting power of members of a committee were proposed by Shapley and Shubik (1954), Penrose (1946) and Banzhaf (1965), and Holler and Packel (1983). We shall refer to them also as SS-power index, PB-power index and HP-power index. There exist also some other well defined power indices, such as Johnston index (1978) and Deegan-Packel index (1979).[1]

In this paper we analyse the Shapley-Shubik, Penrose-Banzhaf and Holler-Packel concepts of power measurement the so called I-power (voter's potential influence over the outcome of voting) and P-power (expected relative share in a fixed prize available to the winning group of committee members) classification introduced by Felsenthal et al. (1998). We show that objections to the Shapley-Shubik index, based on its interpretation as a P-power concept, are not sufficiently justified. The Shapley-Shubik, Penrose-Banzhaf and Holler-Packel measures can all be successfully derived as cooperative game values, and at the same time they can be interpreted as probabilities of being in some decisive position (pivot, swing) without using cooperative game theory at all (Turnovec 2004; Turnovec et al. 2004).

It is demonstrated in the paper that both pivots and swings can be introduced as special cases of a more general concept of decisiveness based on assumption of equi-probable orderings expressing preferences of com-

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[1] For a comprehensive survey and analysis of power indices methodology, see Felsenthal and Machover (1998).

mittee members' support for voted issue. New general a priori voting power measure was proposed in Turnovec (2007), distinguishing between absolute and relative power and covering Shapley-Shubik, Penrose-Banzhaf and Holler-Packel indices as its special cases.

2. Basic Concepts

Let $N = (1,\ldots,n)$ be the set of members (players, parties) and $\omega_i (i = 1,\ldots,n)$ be the (real, non-negative) weight of the *i*-th member such that

$$\sum_{i\in N}\omega_i = \tau, \omega_i \geq 0$$

(e.g. the number of votes of party *i*, or the ownership of *i* in number of shares), where τ is the total weight of all members. Let γ be a real number such that $0 < \gamma \leq \tau$. The $(n+1)$-tuple $[\gamma,\boldsymbol{\omega}] = [\gamma,\omega_1,\ldots,\omega_n]$ such that

$$\sum_{i=1}^{n}\omega_i = \tau, \omega_i \geq 0, 0 < \gamma \leq \tau$$

we call a committee (or a weighted voting body) of the size *n* with quota γ, total weight τ, and allocation of weights $\boldsymbol{\omega}$. Any non-empty subset $S \subseteq N$ we call a voting configuration. Given an allocation ω and a quota γ, we say that $S \subseteq N$ is a winning voting configuration, if $\sum_{i\in S}\omega_i \geq \gamma$ and a losing voting configuration, if $\sum_{i\in S}\omega_i < \gamma$. A configuration *S* is winning, if it has a required majority, otherwise it is losing.

Let

$$G = \left[(\gamma,\boldsymbol{\omega}) \in \mathbb{R}_{n+1} : \sum_{i=1}^{n}\omega_i = \tau, \omega_i \geq 0,\ 0 \leq \gamma \leq \tau\right]$$

be the space of all committees of the size *n*, total weight τ and quota γ. A *power index* is a vector valued function $\Pi : G \rightarrow \mathbb{R}_n$ mapping the space *G* of all committees into $\mathbb{R}_n$. A power index represents for each of the committee members a reasonable expectation that she will be 'decisive' in the sense that her vote (*yes* or *no*) will determine the final outcome of voting. The probability to be decisive we call an absolute power index of an individual member; by normalization of an absolute power index we obtain a relative power of an individual member. By $\Pi_i(\gamma,\boldsymbol{\omega})$ and $\pi_i(\gamma,\boldsymbol{\omega})$ we denote the absolute and relative power an index grants to the *i*-th member of a committee with allocation of weights $\boldsymbol{\omega}$, total weight τ, and quota γ.

Minimal requirements usually imposed on power mapping $\Pi : G \rightarrow \mathbb{R}_n$ are the *anonymity*, *dummy*, and *symmetry* axioms.

Let $[\gamma,\boldsymbol{\omega}]$ be a committee with the set of members *N* and $\sigma : N \rightarrow N$ be a permutation mapping. The committee $[\gamma,\sigma\boldsymbol{\omega}]$ we call a permutation of the committee $[\gamma,\boldsymbol{\omega}]$, $\sigma(i)$ being the new name (number) of the member with original name i. The anonymity axiom requires that $\Pi_i(\gamma,\boldsymbol{\omega}) = \Pi_{\sigma(i)}(\gamma,\sigma\boldsymbol{\omega})$. This says that power is a property of being a committee member and not of the name or number of the committee member.

A member $i \in N$ of a committee $[\gamma,\boldsymbol{\omega}]$ is said to be dummy if she cannot benefit any voting configuration by joining it, i.e. the member i is dummy if $\sum_{k\in S}\omega_k \geq \gamma \Rightarrow \sum_{k\in S\setminus\{i\}}\omega_k \geq \gamma$ for any winning configuration $S \subseteq N$ such that $i \in S$. The dummy axiom requires that $\Pi_i(\gamma,\boldsymbol{\omega}) = 0$ if and only if i is dummy.

Two distinct members i and j of a committee $[\gamma,\boldsymbol{\omega}]$ are called symmetric if their benefit to any voting configuration is the same, i.e. for any S such that $i, j \notin S$ then $\sum_{k\in S\cup i}\omega_k \geq \gamma \Leftrightarrow \sum_{k\in S\cup j}\omega_k \geq \gamma$. This axiom requires that the power of symmetric members is the same.

To define a particular power measure means to identify some qualitative property (decisiveness) whose presence or absence in voting process can be established and quantified (Nurmi 1997). Generally there are two such properties related to the position of committee members in voting that are used as a starting point for quantification of an a priori voting power: a *swing* position and a *pivotal* position of a member.

3. Pivots and Swings

The SS-power index is based on the concept of *pivot*. Let the numbers $1,\ldots,n$ be the fixed names of committee members and $(i_1,\ldots,i_n)$ be a permutation of the members of the committee. Let us assume that member k is in a position r in this permutation, i.e. $k = i_r$. We say that k is in a pivotal position (has a pivot) with respect to a permutation $(i_1,\ldots,i_n)$, if

$$\sum_{j=1}^{r-1}\omega_{i_j} < \gamma \quad \text{and} \quad \sum_{j=1}^{r}\omega_{i_j} \geq \gamma.$$

Assume that a strict ordering of members in a given permutation expresses an intensity of their support (preferences) for a particular issue in the sense that, if a member i_s precedes in this permutation a member i_t, then support by i_s for the particular proposal to be decided is stronger than support by i_t. One can expect that the group supporting the proposal will be formed in the order of positions of members in the given permutation. If it is so, then the k will be in situation when the group composed from preceding members in the given permutation still does not have enough of votes to pass the proposal, and a group of members place behind her in the permutation has not enough of votes to block the proposal. The group that will manage his support will win. A member in a pivotal situation has a decisive influence on the final outcome. Assuming many voting acts and all possible preference orderings equally likely, under the full veil of ignorance about other aspects of individual preferences, it makes sense to evaluate an a priori voting power of each committee member as a probability of being in pivotal situation. This probability is measured by the SS-power index:

$$\Pi_i^{SS}(\gamma,\boldsymbol{\omega}) = \frac{p_i}{n!}.$$

Here p_i is the number of pivotal positions of the committee member i and $n!$ is the number of permutations of the committee members, i.e. number of different orderings of n elements. From $\sum_{i=1}^{n} p_i = n!$ it follows that $\pi_i(\gamma,\boldsymbol{\omega}) = \Pi_i(\gamma,\boldsymbol{\omega})$ (i.e. the relative SS-power index is equal to an absolute one).

The PB-power index is based on the concept of swing. Let S be a winning configuration in a committee $[\gamma,\boldsymbol{\omega}]$. We say that $i \in S$ has a swing in configuration S if $\sum_{k\in S}\omega_k \geq \gamma$ and $\sum_{k\in S\setminus\{i\}}\omega_k < \gamma$. Let s_i denotes the total number of swings of the member i in the committee $[\gamma,\boldsymbol{\omega}]$. The original Penrose definition of voting power was in the absolute form (the absolute PB-power index) given by:

$$\Pi_i^{PB}(\gamma,\boldsymbol{\omega}) = \frac{s_i}{2^{n-1}}.$$

Assuming that all configurations are equally likely this is nothing else but the probability that the given member will be decisive (the probability to have a swing). The PB-power index is frequently used in relative form and is obtained by normalization of the absolute PB-index:

$$\pi_i^{PB}(\gamma,\boldsymbol{\omega}) = \frac{s_i}{\sum\limits_{k\in N} s_k}.$$

The Holler-Packel power index also belongs to the class of swing-based measures. Let S be a winning configuration in a committee $[\gamma,\boldsymbol{\omega}]$. We say that S is a minimal winning configuration if for any $i \in S$ it holds that $\sum_{k\in S}\omega_k \geq \gamma$ and $\sum_{k\in S\setminus\{i\}}\omega_k < \gamma$. The HP-power index assigns to each member of a committee the share of power proportional to the number of swings in minimal winning configurations of which he is a member. Let m_i denote the total number of swings of the member i in minimal winning configurations in the committee, then the Holler-Packel index in relative form is:

$$\pi_i^{HP}(\gamma,\boldsymbol{\omega}) = \frac{m_i}{\sum\limits_{k\in N} m_k}.$$

It is assumed that all winning configurations are possible but only minimal critical winning configurations are being formed to exclude free-riding of the members that cannot influence the outcome of voting. The 'public good' interpretation (the power of each member of a minimal winning configuration is identical with the power of the minimal winning configuration as a whole, power is indivisible) is used to justify the HP index. Although Holler and Packel never presented an absolute form of their index, it is possible to do it following the logic of the PB index. Assuming that only minimal winning configurations will be formed and all of them are equally likely, we obtain the absolute HP-power index as:

$$\Pi_i^{\mathrm{HP}}(\gamma,\boldsymbol{\omega}) = \frac{m_i}{\mu(\gamma,\omega)}$$

where $\mu(\gamma,\boldsymbol{\omega})$ stands for number of minimal winning configurations in committee $[\gamma,\boldsymbol{\omega}]$. The absolute HP-power index gives the probability that given member will have a swing in a minimal winning configuration (ratio of the number of her memberships in minimal winning configurations to the total number of minimal winning configurations).

The SS-power index, PB-power index, and the HP-power index satisfy anonymity, dummy and symmetry axioms.

4. I-Power and P-Power

Felsenthal et al. (1998) introduced the concepts I- and P-power. By I-power they mean:

> … voting power conceived of as a voter's potential influence over the outcome of divisions of the decision making body: whether proposed bills are adopted or blocked. Penrose's approach was clearly based on this notion, and his measure of voting power is a proposed formalization of a priori I-power (Felsenthal and Machover 2004: 9).

By P-power they mean:

> … voting power conceived as a voter's expected relative share in a fixed prize available to the winning coalition under a decision rule, seen in the guise of a simple TU (transferable utility) cooperative game. The Shapley-Shubik approach was evidently based on this notion, and their index is a proposed quantification of a priori P-power (Felsenthal and Machover 2004: 9).

Hence, the fundamental distinction between I-power and P-power is in the fact that the I-power notion takes the outcome to be the immediate one, passage or defeat of the proposed bill, while the P-power view is that passage of the bill is merely the ostensible and proximate outcome of a division; the real and ultimate outcome is the distribution of fixed a purse – the prize of power – among the winners (Felsenthal and Machover, 2004: 9–10). As a conclusion it follows that SS-power index does not measure a priori voting power, but says how to agree on dividing the 'pie' (benefits of victory).

As the major argument of this classification the authors provide a historical observation: Penrose's 1946 paper was either unnoticed or ignored by mainstream – predominantly American – social choice theorists, and Shapley and Shubik's 1954 paper was seen as inaugurating the scientific study of voting power. Because the Shapley-Shubik paper was wholly based on cooperative game theory, it induced among social scientists an almost universal unquestioning belief that the study of power was necessarily and entirely a branch of that theory (Felsenthal and Machover, 2004: 8). The conclusion follows on the grounds that given cooperative game theory with transferable utility is about how to divide a pie, and SS-power index was derived as a special case of Shapley value of cooperative game, the SS-power index is about P-power and does not measure voting power as such.

We demonstrated above that one does not need cooperative game theory to define and justify SS-power index. The SS-power index is a probability to be in a pivotal situation in an intuitively plausible process of forming a winning configurations with no division of benefits involved whatsoever. It is interesting to note that the SS-power index appeared as a special case of Shapley value for cooperative games with the transferable utility, but in exactly the same way one can handle the PB-index. Let us make a short excursion into cooperative game theory.

Let N be the set of players in a cooperative game (cooperation among the players is permitted and the players can form coalitions and transfer utility gained together among themselves) and 2^N the power set of N, i.e. the set of all subsets $S \subseteq N$, called coalitions, including empty coalition. The characteristic function of the game is a mapping $v: 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$. The interpretation of $v: 2^N \to \mathbb{R}$ v is that for any subset $S \subseteq N$ the number $v(S)$ is the value (worth) of the coalition S, in terms how much 'utility' the members of S can divide among themselves in any way that sums to no more than $v(S)$ if they all agree. The characteristic function is said to be super-additive if for any two disjoint subsets $S, T \subseteq N$ we have $v(S \cup T) \geq v(S) + v(T)$ i.e. the worth of the coalition $S \cup T$ is equal to at least the sum of worth of its parts acting separately.

Let us denote cooperative game in characteristic function form by $[N, v]$. The game $[N, v]$ is said to be super-additive if its characteristic function is super-additive. By a value of the game $[N, v]$ we mean a non-negative vector $\varphi(N, v)$ such that $\sum_{i \in N} \varphi_i(N, v) = v(N)$. By $c(i, T) = v(T) - v(T \setminus \{i\})$ we denote the marginal contribution of the player $i \in N$ to the coalition $T \subseteq N$. Then, in an abstract setting, the value $\varphi_i(N, v)$ of the i-th player in the game $[N, v]$ can be defined as a weighted sum of his marginal contributions to all possible coalitions he is a member of: $\varphi_i(v) = \sum_{T \subseteq N, i \in T} \alpha(T) c(i, T)$. Different weights $\alpha(T)$ lead to different values.

Shapley (1953) defined his value by the weights

$$\alpha(T) = \frac{(t-1)!(n-t)!}{n!}$$

where t is the cardinality of T. He proved that it is the only value that satisfies the following three axioms: (i) dummy axiom (see above), (ii) anonymity axiom (see above), and (iii) additivity axiom (sum of two games $[N,v]$ and $[N,u]$ is the value $\varphi(N,v+u) = \varphi(N,v) + \varphi(N,u)$).

As Owen (1982) noticed, the relative PB-index is meaningful for general cooperative games with transferable utilities. One can define Banzhaf value by setting the weights

$$\alpha(T) = \frac{v(N)}{\sum_{k \in N, T \subseteq N} c(k,T)}.$$

Owen demonstrates a certain relation between the Shapley value and Banzhaf value of cooperative game with transferable utilities: both give averages of player's marginal contributions, the difference lies in the weighting coefficients (in the Shapley value coefficients depend on size of coalitions, in the Banzhaf value they are independent of coalition size).

There is also a similar generalization of the Holler-Packel public good index (which is based on membership in minimal winning configurations, in which each member has a swing) as a cooperative game value (Holler and Li 1995).

The relation between values and power indices is straightforward: a cooperative characteristic function game represented by a characteristic function v such that v takes only the values 0 and 1 is called a simple game. With any committee with quota γ and allocation ω we can associate a super-additive simple game such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} \omega_i \geq \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Super-additive simple games can be used as natural models of voting in committees. Shapley and Shubik (1954) applied the concept of the Shapley value for general cooperative characteristic function games to the super-additive simple games as a measure of voting power in committees. Here we demonstrated that PB relative power index can be extended as a value for general cooperative characteristic function games. But we do not need cooperative games to model voting in committees.

5. A Generalized Concept of Decisiveness

In a committee $[\gamma, \boldsymbol{\omega}]$ with the set of members N, let $(S_1, \ldots, S_k), k \leq n$ be a partition of N, i.e. $\bigcup_{j=1}^{k} S_j = N$ and for any $s, t \in N, s \neq t$ it holds that

$S_s \cap S_t = \varnothing$. Let $(j_1,\ldots,j_k)$ be a permutation of numbers $(1,\ldots,k)$, then $S = (S_{j_1},\ldots,S_{j_k})$ we call an ordering defined on *N*, ordered collection of groups of voters.

Considering a particular issue being voted, the following interpretation of an ordering *S* on *N* is possible. Let $r \in S_{j_u}, t \in S_{j_v}$. (a) If $u < v$, then a member *r*'s support for the particular proposal is stronger than the support of member *t*. (b) If $u = v$, then support for a particular proposal by the both r and t is the same.

If $u < v$, it is plausible to assume, that if members of S_{j_v} vote 'yes', then the members of S_{j_u} also vote 'yes', if the members of S_{j_u} vote 'no', then the members from S_{j_v} also vote 'no'. From formal reason we denote $S_{j_o} = \{0\}$ and $\omega_0 = 0$. Let $S = (S_1,\ldots,S_v,\ldots,S_k)$ be an ordering. We say that a group S_v is in a pivotal position in ordering *S*, if

$$\sum_{j=0}^{v-1}\sum_{i\in S_j}\omega_i < \gamma \quad \text{and} \quad \sum_{j=1}^{v}\sum_{i\in S_j}\omega_i \geq \gamma.$$

Under our assumptions, if the pivotal group votes 'yes', then the outcome of voting is 'yes' and if they vote 'no', then the outcome is 'no'.

We say that a member $t \in S_v$ $(v > 0)$ is in a *decisive* situation (has a swing) in an ordering $(S_1,\ldots,S_v,\ldots,S_k)$ if S_v is pivotal group and $S_v \setminus \{t\}$ is not a pivotal group. Let the group S_v is pivotal and $t \in S_v$, then *t* is decisive if and only if either

$$\sum_{j=0}^{v-1}\sum_{i\in S_j}\omega_i + \sum_{i\in S_v \setminus \{t\}}\omega_i < \gamma$$

(i.e. if the group S_v joins preceding groups voting 'yes', then by changing unilaterally his 'yes' to 'no' member *t* changes the outcome of voting from 'yes' to 'no'), or

$$\sum_{j=0}^{v-1}\sum_{i\in S_j}\omega_i + \omega_t \geq \gamma$$

(i.e. if the group S_v does not join preceding groups voting 'yes', then by changing unilaterally his 'no' to 'yes' member *t* changes the outcome of voting from 'no' to 'yes'). The generalized concept of decisiveness combines the logic of pivots and swings: member of a committee is decisive, if she has swing in a pivotal group.

6. Partition and Ordering Numbers

Let *N* be a finite set of size *n*, *k* a positive integer $(1 \leq k \leq n)$, and $T(N,k) = (T_1,T_2,\ldots,T_k)$ be a set of *k* disjoint nonempty subsets of *N* such that for any $r,s \leq k, r \neq s, T_r \cap T_s = \varnothing, \bigcup_{j=1}^{k} T_j = N$ (partition of *N* of the size *k*). Let

Table 1. Partition Numbers $p(n,k)$, Stirling's Numbers of the Second Kind

n/k	1	2	3	4	5	6	7	8	Total
1	1								1
2	1	1							2
3	1	3	1						5
4	1	7	6	1					15
5	1	15	25	10	1				52
6	1	31	90	65	15	1			203
7	1	63	301	350	140	21	1		877
8	1	127	966	1701	1050	266	28	1	4140

Table 2. Ordering Numbers $o(n,k)$

n/k	1	2	3	4	5	6	7	8	Total
1	1								1
2	1	2							3
3	1	6	6						13
4	1	14	36	24					75
5	1	30	150	240	120				541
6	1	62	540	1560	1800	720			4683
7	1	126	1806	8400	16800	15120	5040		47293
8	1	254	5796	40824	126000	191520	141120	40320	545835

$P(N,k)$ be the set of all partitions of N of the size k, and $p(n,k)$ the cardinality of $P(N,k)$. Then, setting $p(n,k)=1$, and $p(n,n)=1$ for any positive integers k and n such that $1<k<n$ we have:

$$p(n,k)=p(n-1,k-1)+kp(n-1,k).$$

The above recurrence relation generates so called Stirling's numbers of the second kind (we shall refer to them also as partition numbers), which count the number of ways to partition a set of n elements into k nonempty subsets, see e.g. Hall (1967). In Table 1 we provide several values of Stirling's numbers of the second kind.

For instance, the number $p(6,4)=65$ in column $k=4$ and row $n=6$, indicating that there exist 65 distinct partitions of 6 elements into 4 nonempty subsets, is given by $65=25+(4\times 10)$, where 25 is the number above and to the left of 65, 10 is the number above 65, and 4 is the number of column containing the 10. The last column provides total number of partitions of n elements. There exists also an explicit formula for partition numbers:

$$p(n,k)=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{n}.$$

Having a partition of N of the size k, there exist $k!$ ways how to order its elements. Let us denote by $O(N,k)$ the set of all orderings of the size k defined on N, and $o(n,k)$ its cardinality. Then, setting $o(n,1)=1$ and $o(n,n)=n!$ for any positive integers k and n such that $1<k<n$, we have $o(n,k)=ko(n-1,k-1)+ko(n-1,k)$. This recurrence relation directly follows from Stirling's numbers of the second kind. We shall refer to $o(n,k)$ as ordering numbers.

In Table 2 we provide several values of ordering numbers. For instance, the number $o(5,3)=150$ in column 3 and row $n=3$, indicating that there exist 150 distinct orderings of 5 elements with 3 nonempty indifference classes, is given by $150=3\times 14+3\times 36$, where 14 is the number above and to the left of 150, 36 is the number above 150, and 3 is the number of column containing the 150. The last column provides total number of orderings of n elements. There exists also an explicit formula for ordering numbers:

$$o(n,k)=\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{n}.$$

7. Generalized Power Indices

Using definition of decisiveness from section 5, it makes sense to measure a priori voting power of the member of a committee by number of her decisive situations. Assuming that in sufficiently large number of voting acts all orderings (expressing preferences of members' groups) are equiprobable, then measure of so called absolute power of the member i is given by the ratio of the number of her decisive situations to the number of orderings, and relative power is given by the ratio of the number of decisive situations of the member i to the total number of decisive situations.

In a committee $[\gamma,\omega]$ with set of members N, let $O(N)=\bigcup_{k=1}^{n}O(n,k)$ be a set of all orderings, including the weak ones. By $O(n,n)$ we denote the set of all strict orderings, and by $O(n,2)$ the set of all binary orderings. To follow standard logic of voting, we extend the set of binary orderings by two orderings: $(\varnothing,N)$ and $(N,\varnothing)$, reflecting two bipartitions of 'no' and 'yes' voters (everybody prefers vote 'no', everybody prefers vote 'yes'). Let us denote extended set of binary orderings by

$$B(N)=O(n,2)\cup\{(\varnothing,N),(N,\varnothing)\}.$$

Then, for evaluation of a priori voting power, we consider generic set of orderings

$$W(N)=\bigcup_{k=2}^{n}O(n,k)\cup\{(\varnothing,N),(N,\varnothing)\}.$$

Clearly, $|(S(N))| = n!$ and $|(B(N))| = 2^n$ and

$$|(W(N))| = \sum_{k=2}^{n} o(n,k) + 2 = \sum_{k=2}^{n}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n + 2.$$

Let us define

$$\rho_i(O) = \begin{cases} 1 & \text{if } i \text{ is decisive in } O \\ 0 & \text{otherwise} \end{cases}$$

where $O \in W(N)$, and $\theta(O)$ such that

$$\theta(O) \geq 0, \sum\nolimits_{O \in W(N)} \theta(O) = 1$$

which is a probability that ordering O will appear. Then,

$$\Pi_{i,\theta}(\gamma,\boldsymbol{\omega}) = \sum_{O \in W(N)} \theta(O)\rho_i(O)$$

is a probability that, given a probability distribution θ over the set $W(N)$, $i \in N$ will be decisive in a committee $[\gamma,\boldsymbol{\omega}]$, an absolute general form of power index. The general form of a relative power index of the member i is:

$$\pi_{i,\theta}(\gamma,\boldsymbol{\omega}) = \frac{\sum_{O \in W(N)} \theta(O)\rho_i(O)}{\sum_{k \in N}\sum_{O \in W(N)} \theta(O)\rho_k(O)}$$

which is the share of the i-th member in total power.

Considering strict orderings only and selecting

$$\theta^{SS}(O) = \begin{cases} \dfrac{1}{|S(N)|} = \dfrac{1}{n!} & \text{if } O \in S(N) \\ 0 & \text{otherwise} \end{cases}$$

which is the equi-probability of strict orderings and from which we obtain the absolute SS-power index. From $\sum_{i \in N}\sum_{O \in S(N)} \theta^{SS}(O)\rho_i(O) = n!$ it follows that absolute SS-power index is equal to relative SS-power index.

Considering the extended set of binary orderings only and selecting

$$\theta^{PB}(O) = \begin{cases} \dfrac{1}{|B(N)|} = \dfrac{1}{2^n} & \text{if } O \in B(N) \\ 0 & \text{otherwise} \end{cases}$$

which is the equi-probability of binary orderings, we obtain the absolute PB-power index: the ratio of number of swings to the number of binary orderings.

The framework can also be used to define the Holler-Packel index. Let

$D(N) \subseteq B(N)$ be a set of such binary orderings in which each member of pivotal group has swing (orderings with minimal pivotal groups). Considering only such binary orderings and selecting

$$\theta^{HP}(O) = \begin{cases} \dfrac{1}{|D(N)|} & \text{if } O \in D(N) \\ 0 & \text{otherwise} \end{cases}$$

we obtain the absolute and relative versions of this index.

Considering all orderings, including weak ones, and selecting

$$\theta^{GW}(O) = \begin{cases} \dfrac{1}{|(W(N)|} = \dfrac{1}{\sum_{k=2}^{n}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^{n} + 2} & \text{if } S \in S(N) \\ 0 & \text{otherwise} \end{cases}$$

which is the equi-probability of all orderings, we obtain the absolute General Weak ordering index (GW-power index).

It is easy to prove that for all power indices defined above (based on a generalized concept of decisiveness and the equi-probability of relevant orderings) satisfy the anonymity, symmetry, and dummy axioms defined above.

8. Illustrative Example

To illustrate the concepts introduced above let us use a simple example. Let $N = \{1,2,3\}$. Consider a committee $[51;50,30,20]$. In this case set $W(N) = S(N) \cup B(N)$ consists of 14 orderings: (a) The set of strict orderings $S(N)$ consisting of six orderings $(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)$; and (b) the extended set of binary orderings $B(N)$ consisting of eight binary orderings. Table 3 provides list of orderings and decisive positions. Decisive members in a given ordering by an asterisk. Column σ_i denotes the decisiveness of the i-th member in a given ordering (1 for decisive, 0 for non decisive). Assuming equi-probable orderings we obtain:

GW-index
Absolute: $\Pi_1^{\mathrm{GW}} = \frac{10}{14}, \Pi_2^{\mathrm{GW}} = \frac{3}{14}, \Pi_3^{\mathrm{GW}} = \frac{3}{14}$.
Relative: $\pi_1^{\mathrm{GW}} = \frac{10}{16}, \pi_2^{\mathrm{GW}} = \frac{3}{16}, \pi_3^{\mathrm{GW}} = \frac{3}{16}$.

Shapley-Shubik index (strict orderings only)
Absolute/relative: $\Pi_1^{\mathrm{SS}} = \pi_1^{\mathrm{SS}} = \frac{4}{6}, \Pi_2^{\mathrm{SS}} = \pi_2^{\mathrm{SS}} = \frac{4}{6}, \Pi_3^{\mathrm{SS}} = \pi_3^{\mathrm{SS}} = \frac{1}{6}$.

Penrose-Banzhaf index (binary orderings)
Absolute: $\Pi_1^{\mathrm{PB}} = \frac{6}{8}, \Pi_2^{\mathrm{PB}} = \frac{2}{8}, \Pi_3^{\mathrm{PB}} = \frac{2}{8}$.
Relative: $\pi_1^{\mathrm{PB}} = \frac{6}{10}, \pi_2^{\mathrm{PB}} = \frac{2}{10}, \pi_3^{\mathrm{PB}} = \frac{2}{10}$.

Table 3. List of Orderings and Decisive Situations for the Committee [51;50,30,20]

Orderings	σ_1	σ_2	σ_3	Sum	θ^{SS}	θ^{PB}	θ^{HP}	θ^{GW}
$(1,2^*,3)$	0	1	0	1	$\frac{1}{6}$	0	0	$\frac{1}{14}$
$(1,3^*,2)$	0	0	1	1	$\frac{1}{6}$	0	0	$\frac{1}{14}$
$(2,1^*,3)$	1	0	0	1	$\frac{1}{6}$	0	0	$\frac{1}{14}$
$(2,3^*,1)$	1	0	0	1	$\frac{1}{6}$	0	0	$\frac{1}{14}$
$(3,1^*,2)$	1	0	0	1	$\frac{1}{6}$	0	0	$\frac{1}{14}$
$(3,2,1^*)$	1	0	0	0	$\frac{1}{6}$	0	0	$\frac{1}{14}$
$(1,(2^*,3^*))$	0	1	1	2	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{14}$
$(2,(1^*,3))$	1	0	0	1	0	$\frac{1}{8}$	0	$\frac{1}{14}$
$(3,(1^*,2))$	1	0	0	1	0	$\frac{1}{8}$	0	$\frac{1}{14}$
$((1^*,2^*),3)$	1	1	0	2	0	$\frac{1}{8}$	$\frac{1}{14}$	$\frac{1}{14}$
$((2,3),1^*)$	1	0	0	1	0	$\frac{1}{8}$	$\frac{1}{14}$	$\frac{1}{14}$
$((1^*,3^*),2)$	1	0	1	2	0	$\frac{1}{8}$	$\frac{1}{14}$	$\frac{1}{14}$
$((1^*,2,3))$	1	0	0	1	0	$\frac{1}{8}$	0	$\frac{1}{14}$
$((1,2,3))$	0	0	0	0	0	$\frac{1}{8}$	0	$\frac{1}{14}$
for GW	10	3	3	16				
for SS	4	1	1	6				
for PB	6	2	2	10				
for HP	3	2	2	7				

Holler-Packel power index (binary orderings with minimal pivotal groups)
Absolute: $\Pi_1^{HP} = \frac{3}{4}, \Pi_2^{HP} = \frac{2}{4}, \Pi_3^{HP} = \frac{2}{4}$.
Relative: $\pi_1^{HP} = \frac{3}{7}, \pi_2^{HP} = \frac{2}{7}, \pi_3^{HP} = \frac{2}{7}$.

Note that while the SS-power index and PB-power index based on generalized concept of decisiveness provides the same values as the original definitions, it is not the case for HP-index. For instance, the original definition the HP-power index (relative form) is in our case $\left(\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\right)$, but we obtained $\left(\frac{3}{7}, \frac{2}{7}, \frac{2}{7}\right)$. This follows from the difference in concepts of minimal winning configurations and minimal pivotal groups. The set of binary orderings with minimal winning configurations is a subset of the set of binary orderings with minimal pivotal groups, and permutation of the same binary partition does not necessarily provide the same decisiveness to the same committee members.

9. Concluding Remarks

The primary purpose of this paper was not to introduce a new power index, but rather to demonstrate that there is no contradiction between pivot-based and swing-based measures of voting power: there is no fundamental distinction between pivots and swings. Both concepts appear to be special cases of a more general concept of decisiveness based on full range of possible preferences of individual committee members and their groups; and both follow from the same logic and can be formulated in the same framework. Moreover, we do not need cooperative game theory to define and analyze a priori voting power. In a sense a game-theoretical setting of the problem restricts the set of analytical tools.

Nevertheless, it might be of some interest to investigate more deeply properties of generalized concept of power, namely its monotonicity properties: local monotonicity (the member with greater weight cannot have less power than the member with smaller weight) and global monotonicity (if the weight of one member is increasing and the weights of all other members are decreasing or staying the same, then the power of the 'growing weight' member will at least not decrease) of absolute and relative forms of power measures. Reconsideration of other power indices (Johnston, Deegan-Packel, spatial modifications of power indices) in the framework of generalized concept of decisiveness might contribute to deeper understanding of quantification of voting power. Also an application of this approach to transferable utility cooperative games and its values can bring new results.

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