On the equity principle and the balanced solution for cooperative TU-games[1]

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Abstract:

The Shapley value of a cooperative transferable utility game distributes the dividend of each coalition in the game equally among its members. Given exogenous weights for all players, the corresponding weighted Shapley value distributes the dividends proportionally to their weights. In this contribution we define the *balanced solution* which assigns weights to players such that the corresponding weighted Shapley value of each player is equal to her weight. We prove its existence for all monotone transferable utility games, discuss properties of this solution, and provide an axiomatization using balanced transformations.

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1. Introduction

Consider a finite set N with n players. Situations where each subset of players of N can generate a certain worth can be described by a *cooperative transferable utility game* (or simply *TU-game*) (N, v): for any *coalition* $S \subseteq N$, the real number $v(S)$ is the *worth* of S, which the members of the coalition S can distribute among themselves. Note that $v(S)$ can be also interpreted as the *cost* which has to be splitted among members of the coalition S. A *payoff vector* in an n-player TU-game is an n-dimensional vector whose components are the payoffs of the corresponding players. A single-valued *solution* for a class $\mathcal{C}$ of TU-games is a function that assigns a payoff vector to every TU-game in $\mathcal{C}$. The best known single-valued solution for TU-games is the *Shapley value* (Shapley, 1953b) which distributes the so-called *Harsanyi dividends* of the game *equally* among the players in the corresponding coalitions (see Section 2 for undefined notions).

The equal distribution of the dividends seems questionable in situations that suggest proportionality rather than equality. The standard business practice of dividing a firm's profit proportionally to investment (constant return per share) could serve as a daily-life example of such a situation. (Subadditive) cost games provide another example. Consider a market with non-linear pricing where the unit price depends on the purchased volume, e.g., it equals 10 for quantities below four and 8 for higher amounts. Suppose there are two agents in the market who want to buy two and three units, respectively. The problem can be described by the following TU-game

$$N = \{1, 2\}, \quad v(\{1\}) = 20, \quad v(\{2\}) = 30, \quad v(\{1, 2\}) = 40.$$

If the buyers agree to pool their resources and buy together the five units to guarantee the lower price for both of them, it seems reasonable that they spend 16 and 24 units, respectively, and pay the same unit price. On the contrary, all standard solutions for TU-games (Shapley value, nucleolus etc.) are based on equal split of the dividends for two-player games. In this example these solutions predict that buyers split the cost savings (of 10) equally (spending 15 and 25 respectively), and facing different prices of 15/2 and 25/3.

The question "How to split the dividends?," or, in general terms, "How to distribute the benefits or costs among the members of the group?" has been analyzed extensively in the social psychology literature. Homans (1961) and Selten (1978) stated the *equity principle* as a norm of distributive justice. Selten (1978) claims that "Only in special cases does the application of the equity principle give rise to an equal distribution. In many situations there are good reasons for an uneven split." He proposes dividing the benefits (or costs) in proportion to some key numbers, that are *relevant* and *accessible* in a given situation. Although equity theory does not determine the standard of

comparison (i.e., the specific selection of the key numbers) in a general situation, in the simple model of a two-player game we do not face any ambiguity. The only characteristic of a particular agent i is her individual worth $v(\{i\})$. Once the two agents form the grand coalition, the equity principle requires that they share its worth proportionally to their individual worths. This also corresponds to our intuition in the cost-game example above, where the equity principle spreads the expenses between the agents in proportion 2:3, i.e., they will pay 16 and 24, respectively.

Several authors have proposed 'proportional' solutions for particular classes of games (see, e.g., Kalai, 1977; Roth, 1979; Hart and Mas-Colell, 1989; Feldman, 1999; Ortmann, 2000). In this paper we focus on cooperative TU-games. Our approach is based on the *weighted Shapley value*, Shapley (1953a), where the dividends are distributed among players proportionally according to some *exogenously* given weights. Our aim is to *endogenize* these weights to capture the structure of the given TU-game. For two-player TU-games we follow the equity principle and employ the individual worths $v(\{i\})$ as the weights. Formally, we insist on proportional standardness for two-player games (Ortmann, 2000) rather than standardness for two-player games as defined by Hart and Mas-Colell (1988, 1989).

Of course, such a solution will not be covariant under fixed-vector addition, since for two-player TU-games covariance under fixed-vector addition is equivalent to standardness for two-player games. We find standardness for two-player games relevant in situations where players can freely manipulate their individual worths $v(\{i\})$. In such games, $v(\{i\})$ of the particular player is therefore irrelevant, and the equal split of the dividend as proposed by standardness for two-player games is the only natural distribution. In contrast, we focus on situations where individual worths cannot be misrepresented, or the misrepresentation of $v(\{i\})$ is costly for player i.

Example 1. Consider two investors with endowments $e_1 < E$ and $e_2 < E$, respectively, that are faced with an investment opportunity in a market with interest rate r for amounts below E, and $R > r$ for larger amounts. Suppose $e_1 + e_2 > E$. In the underlying TU-game the first investor receives $v(\{1\}) = re_1$, on its own, the second investor $v(\{2\}) = re_2$, and their coalition $v(N) = R(e_1 + e_2)$. In such a game the misrepresentation of $v(\{i\})$ is costly, since the only way to increase $v(\{i\})$ is to borrow at the market rate. Therefore the worths $v(\{i\})$ are informative, they are both relevant and accessible, and applying equity theory they should serve as the key numbers (equity standards).

In this situation, the proportional split with respect to individual worths follows the natural intuition that investors will agree to pool their endowments to guarantee the higher interest rate R for all their funds and to split the investment return proportionally to e_1 and e_2. Formally

$x_i = Re_i$ for both players. The Shapley value, on the contrary, splits the surplus equally, i.e. $x_i = re_i + \frac{1}{2}(R - r)(e_i + e_j)$. Denoting the 'actual individual' interest rate by $r_i = \frac{x_i}{e_i}$ we get for the difference of the individual actual interest rates

$$r_1 - r_2 = \frac{1}{2}\left(\frac{e_2}{e_1} - \frac{e_1}{e_2}\right)(R - r).$$

We can directly see that for $e_1 > e_2$ the Shapley value assigns the 'individual' interest rates $r_1 < r_2$. In consequence in all asymmetric two-player games of this particular class, the Shapley value assigns a smaller interest rate to the 'bigger' investor than to the 'smaller' one.

This feature is not specific for the Shapley value. All solution concepts based on covariance under fixed-vector addition and symmetry reveal this 'reverse' relation between investment endowments and actual interest rate. Therefore, for our example, we offer an alternative 'standard' concept – the invariance to interest irrelevant investment.

Assume that player 1 increases her investment by ε with $e_1 + \varepsilon < E$. This decision should not change the return of player 2 at all, there is no reason to appreciate player 2, she did not add any extra investment to the project. Again, all the solutions based on covariance to fixed vector addition increase the payoff to player 2 by $\frac{\varepsilon}{2}(R - r)$, whereas the payoff to the player 2 based on proportionality is immune to the additional investment of player 1 and remains Re_2.

Since the singleton worths are the only available values characterizing the particular player in a two-player TU-game, the proportionality principle for two-player games is straightforward. However, for games with more than two players all coalition worths are relevant values distinguishing the different players, and the proportionality principle in more than two player games is less obvious. To apply the proportionality principle to a general game we modify the so-called *proper Shapley value* proposed by Vorob'ev and Liapounov (1998). The proper Shapley value distributes the worth of the grand coalition N among the players in such a way that the payoff vector x coincides with the weighted Shapley value with respect to the weight scheme given by that vector x. Thus the proper Shapley value is obtained as a fixed point of the mapping that appears in the definition of the weighted Shapley value.

Vorob'ev and Liapounov (1998) proved the existence of the proper Shapley value for all games with nonnegative dividends. These games form a proper subclass of monotone, convex games. Similarly to their approach, we consider fixed points of a particular mapping on the payoff simplex. Our mapping coincides with that of Vorob'ev and Liapounov for positive efficient weights. For weights on the boundary of the nonnegative efficient simplex (where the weighted Shapley value is

not defined) we follow the original Shapley value approach and split the dividends equally among the players in the corresponding coalition.

The idea behind our solution is straightforward. The weighted Shapley value re-distributes dividends with respect to given exogenous weights to produce a 'new' vector of weights. In our opinion, only vectors of weights that are self-enforcing are consistent with the re-distribution mechanism. We call the stable payoff vectors that are invariant to the re-distribution of dividends *balanced values*. The *balanced solution* assigns to each game the set of all balanced values.

The paper is organized as follows. In Section 2 we present some basic facts about TU-games and their solutions. The balanced value and the balanced solution are introduced in Section 3. We discuss some basic properties and show that each monotone game admits at least one balanced value. In Section 4 we provide an axiomatization of the balanced solution. Finally, proofs of the main results are presented in Section 5.

2. Preliminaries

Let us start with several formal definitions. A *transferable utility game* (*TU-game* for short) is a pair (N, v) where $N = \{1, \ldots, n\}$ and v is a *characteristic function* assigning to each subset $S \subseteq N$ a real number $v(S)$ with $v(\emptyset) = 0$. We denote the collection of all TU-games by $\mathcal{G}$.

A TU-game (N, v) is

- *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$,
- *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ whenever $S, T \subseteq N$ are disjoint.

Let $(N, v) \in \mathcal{G}$. The *dividends* $\Delta_{N,v}(S)$, $S \subseteq N$, are defined inductively by

$$\Delta_{N,v}(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ v(S) - \sum_{T \subsetneq S} \Delta_{N,v}(T), & \text{if } S \neq \emptyset \end{cases}$$

(see Harsanyi, 1959). Let us note that $v(S) = \sum_{T \subseteq S} \Delta_{N,v}(T)$ for every $S \subseteq N$. This formula shows that the dividends uniquely determine the characteristic function.

We employ the following notation. Let N be a finite nonempty set, $y \in \mathbb{R}^N$, and $S \subseteq N$. The symbol $y|_S$ stands for the restriction of y to S, and y_S stands for $\sum_{i \in S} y_i$ with $y_\emptyset = 0$.

A payoff vector $x \in \mathbb{R}^N$ for a game (N, v) is *efficient* if it exactly distributes the worth $v(N)$ of the grand coalition N, i.e., if $x_N = v(N)$. The set of all efficient payoff vectors of (N, v) is denoted by $X(N, v)$ and the set of all efficient payoff vectors with positive coordinates is denoted by $X_+(N, v)$. If there is no danger of confusion we write simply X and X_+ instead of $X(N, v)$ and $X_+(N, v)$.

Let $\mathcal{C} \subseteq \mathcal{G}$ be a (sub)class of games on N. A *single-valued solution* on $\mathcal{C}$ is a function f that assigns to every game $(N, v) \in \mathcal{C}$ a payoff vector $f(N, v)$. A single-valued solution f is *efficient* on $\mathcal{C}$ if $f(N, v)$ is an efficient payoff vector for all $(N, v) \in \mathcal{C}$. A *set-valued solution* F on $\mathcal{C}$ assigns a set of payoff vectors $F(N, v)$ to every game $(N, v) \in \mathcal{C}$. A set-valued solution F is *efficient* on $\mathcal{C}$ if every payoff vector in $F(N, v)$ is efficient whenever $(N, v) \in \mathcal{C}$.

The *Shapley value* (Shapley, 1953b) of a game (N, v) is an efficient single-valued solution obtained by distributing the dividends of every coalition equally among all players in the coalition, i.e., it is the function $\varphi : \mathcal{G} \to \mathbb{R}^N$ defined by

$$\varphi(N, v) = (\varphi_i(N, v))_{i \in N}, \qquad \text{where}$$

$$\varphi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{1}{|S|} \Delta_{N,v}(S), \quad i \in N. \tag{1}$$

(The symbol $|S|$ denotes the cardinality of S.)

Given a weight vector $\omega \in \mathbb{R}^N$ with positive weights $\omega_i > 0$, $i \in N$, the corresponding *weighted Shapley value* (Shapley, 1953a) is the function $\varphi^\omega : \mathcal{G} \to \mathbb{R}^N$ defined by

$$\varphi_i^\omega(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\omega_i}{\omega_S} \Delta_{N,v}(S), \quad i \in N.$$

The weighted Shapley value thus distributes the dividends of coalitions proportionally to the exogenously given weights of the players. Clearly, if all weights ω_i are equal to each other then the weighted Shapely value $\varphi^\omega(N, v)$ is equal to the Shapley value $\varphi(N, v)$. Further observe that if ω and $\widetilde{\omega}$ are positive weight vectors with $\widetilde{\omega}_i / \widetilde{\omega}_j = \omega_i / \omega_j$ for all $i, j \in N$ then $\varphi^\omega(N, v) = \varphi^{\tilde{\omega}}(N, v)$.

The best known set-valued solution is the *core* which assigns to every game the set of efficient payoff vectors that are group stable in the sense that every coalition gets at least its own worth. So, the core of a game (N, v) is the set of payoff vectors given by

$$\text{Core}(N, v) = \{x \in \mathbb{R}^N \mid x_N = v(N) \text{ and } x_S \geq v(S) \text{ for all } S \subseteq N\}.$$

Of course, core can be empty, even for monotone superadditive games.

Let us recall several notions which will be helpful later.

Definition 1. Let (N, v) be a TU-game.

- A *component* in (N, v) is a coalition $C \subseteq N$ such that

$$v(S) = v(S \setminus C) + v(S \cap C)$$

for all $S \subseteq N$ (see, e.g., Aumann and Drèze, 1980; Chang and Kan, 1994).

- Player $i \in N$ is a *null player* in (N, v) if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$.

- Players $i, j \in N$ are *symmetric* in (N, v) if, for each $S \subseteq N \setminus \{i, j\}$, we have $v(S \cup \{i\}) = v(S \cup \{j\})$.

In the next definition we recall some notions related to solutions of TU-games.

Definition 2. Let $\mathcal{C} \subseteq \mathcal{G}$ be a class of games and F be a solution defined on $\mathcal{C}$. A solution F satisfies on $\mathcal{C}$

- *component efficiency* if, for every $(N, v) \in \mathcal{C}$, $x \in F(N, v)$, and every component C in (N, v), we have $x_C = v(C)$;

- the *null player property* if, for every $(N, v) \in \mathcal{C}$ and $x \in F(N, v)$, we have $x_i = 0$ whenever i is a null player in the game (N, v);

- *local monotonicity* if, for every $(N, v) \in \mathcal{C}$, $i, j \in N$, and $x \in F(N, v)$, we have $x_i \geq x_j$ whenever i and j satisfy $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$;

- the *symmetry property* if, for every $(N, v) \in \mathcal{C}$ and $x \in F(N, v)$, we have $x_i = x_j$ whenever i and j are symmetric players in (N, v);

- *individual rationality*, if $x_i \geq v(\{i\})$ for every $(N, v) \in \mathcal{C}$, $x \in F(N, v)$, and $i \in N$.

3. BALANCED VALUES AND THE BALANCED SOLUTION

Let (N, v) be a given game and $\omega^1 \in X_+$ be an initial weight scheme where the weights equal to each other. Applying the Shapley value (1) we get a redistributed weight (or payoff) vector $\omega^2 := \varphi(N, v)$ reflecting the power of the players. In correspondence, we identify ω^2 with a new weight scheme for players in the game (N, v) and apply the weighted Shapley value with these weights. We obtain $\omega^3 = \varphi^{\omega^2}(N, v)$. Applying the weighted Shapley value repeatedly we get a sequence $(\omega^k)_{k=1}^{\infty}$ of weights satisfying $\omega^{k+1} = \varphi^{\omega^k}(N, v)$ (assuming ω^k to have strictly positive coordinates). Obviously, the 'limit weights' (if such exist) will be invariant to the redistribution mechanism. This turns our attention to the fixed points of the mapping $\omega \mapsto \varphi^{\omega}(N, v)$.

Let us be more formal. For $(N, v) \in \mathcal{G}$, define $h(N, v) : X \to \mathbb{R}^N$ by

$$h(N, v)_i(x) = \sum_{\substack{S \subseteq N, i \in S \\ x_S \neq 0}} \frac{x_i}{x_S} \Delta_{N, v}(S) + \sum_{\substack{S \subseteq N, i \in S \\ x_S = 0}} \frac{1}{|S|} \Delta_{N, v}(S), \quad i \in N.$$

For the sake of brevity, we omit the parameters (N, v) if no confusion is possible.

The mapping h coincides with the mapping $\omega \mapsto \varphi^{\omega}(N, v)$ on X_+. The second sum in the definition of h is important when dealing with zero weights. Since we cannot divide by x_S if $x_S = 0$,

in that case we follow the original Shapley value approach and split the dividends equally among the players in the corresponding coalition.

Remark 1. For every $x \in X$ we have

$$h_N(x) = \sum_{i \in N} h_i(x) = \sum_{\substack{S \subseteq N \\ x_S \neq 0}} \Delta_{N,v}(S) + \sum_{\substack{S \subseteq N \\ x_S = 0}} \Delta_{N,v}(S) = \sum_{S \subseteq N} \Delta_{N,v}(S)$$

$$= v(N),$$

implying that h maps values from X to X.

The next definition introduces the key notion of our paper.

Definition 3. Let $(N, v) \in \mathcal{G}$. A vector $x \in X$ is called a *balanced value* of (N, v) if $h(x) = x$ and $x_i \geq 0$ for all $i \in N$. We denote

$$B(N, v) = \{x \in X(N, v) \mid x \text{ is a balanced value of } (N, v)\},$$

as the set of balanced values of game (N, v). We refer to the solution that assigns to every game $(N, v) \in \mathcal{G}$ the set of all balanced values $B(N, v)$ as the *balanced solution*.

Remark 2. (i) We require the balanced values to be nonnegative since we consider them as payoff vectors as well as weight schemes.

(ii) Consider a two-player game (N, v), where $N = \{1, 2\}$ and v satisfies $v(\{1\}) > 0$, $v(\{2\}) > 0$, and $v(N) > 0$. Then a straightforward computation gives a (unique) balanced value

$$\left(\frac{v(\{1\})}{v(\{1\}) + v(\{2\})} \cdot v(N), \frac{v(\{2\})}{v(\{1\}) + v(\{2\})} \cdot v(N) \right).$$

Thus in this case the worth of the grand coalition is distributed *proportionally* to the individual worths.

(iii) In general, fixed points for $h : X \to X$ need not exist, as one can easily check in the case of the following two-player game:

$$N = \{1, 2\}, \quad v(\{1\}) = 1, \quad v(\{2\}) = -1, \quad v(N) = 1.$$

However, we have the following first main result.

Theorem 1. *There exists at least one balanced value for each monotone game.*

Let us indicate here the main idea of the proof. The complete proof can be found in Section 5.

The first step is to prove that the function h redistributes each positive weight scheme $x \in X_+$ to a positive weight scheme $h(x) \in X_+$. This statement is self-evident for games with positive dividends.

We prove it for general monotone games in Lemma 3(ii). Second, we define a multivalued mapping F by assigning to each x from the closure of X_+ the convex hull of the set of all limit values of $h|_{X_+}$ at x. Third, applying Kakutani's fixed point theorem to F, we get a fixed point x^* of F. Finally, we show that x^* is also a fixed point of h.

Note that we cannot apply directly Brouwer's fixed point theorem since the mapping h is not continuous on X, as illustrated in the following example.

Example 2. Consider the monotone game (N, v) given by $N = \{1, 2, 3\}$ with the dividends

$$\Delta_{N,v}(S) = \begin{cases} 1 & \text{if } S \in \{\{3\}, \{1, 2\}\}, \\ 0 & \text{otherwise;} \end{cases}$$

and vectors $x^\varepsilon = (\varepsilon, 2\varepsilon, 2 - 3\varepsilon)$. Clearly, $h(x^\varepsilon) = (1/3, 2/3, 1)$ whenever $\varepsilon \in (0, 2/3]$, but $h(x^0) = (1/2, 1/2, 1)$.

For monotone superadditive games, individual rationality seems a desirable property. It turns out that this property is satisfied by the balanced solution.

Theorem 2. *If (N, v) is a monotone superadditive game and x is a balanced value of (N, v), then $x_i \geq v(\{i\})$ for every $i \in N$.*

The proof of this theorem can also be found in Section 5.

Balanced values are not unique, even for the class of monotone games, as can be seen from the game (N, v) given by $v(S) = 0$ for every $S \subsetneq N$ and $v(N) = 1$, where each $x \in X_+$ is a balanced value. On the other hand, we do not know whether the balanced value is determined uniquely for any monotone or superadditive game (N, w) with $w(\{i\}) > 0$ for every $i \in N$.

The next proposition captures properties of balanced values of monotone games related to the symmetry property, the null player property and local monotonicity.

Proposition 1. *Let (N, v) be a monotone game.*

    (i) *If $i, j \in N$, $i \neq j$, are symmetric players in (N, v), then there exists a balanced value x^* of (N, v) with $x_i^* = x_j^*$.*

    (ii) *Let $i \in N$. If $v(\{i\}) = 0$, then there exists a balanced value x^* of (N, v) with $x_i^* = 0$.*

    (iii) *Let $i \in N$. If $v(\{i\}) > 0$, then each balanced value x^* of (N, v) satisfies $x_i^* > 0$.*

    (iv) *Let $k, l \in N$, $k \neq l$, satisfy $v(\{k\}) > 0$ and $v(\{l\}) > 0$. If $v(S \cup \{k\}) \geq v(S \cup \{l\})$ for every $S \subseteq N \setminus \{k, l\}$, then each balanced value x^* of (N, v) satisfies $x_k^* \geq x_l^*$.*

(v) *If $i \in N$ is a null player in (N, v), then $x_i^* = 0$ for every balanced value x^* of (N, v).*

The proof of this proposition also can be found in Section 5. As a corollary of part (iv) we immediately obtain the following.

Corollary 1. *Let (N, v) be a monotone game. If $k, l \in N$ are symmetric players with respect to (N, v) and $v(\{k\}) > 0$ (and thus $v(\{l\}) > 0$) then each balanced value x^* of (N, v) satisfies $x_k^* = x_l^*$.*

It is well known that efficiency, the symmetry property, the null player property, and additivity uniquely determine the Shapley value. The balanced solution satisfies efficiency and the null player property on the class of all games, and local monotonicity and consequently the symmetry property on the class of all positive monotone games. (A game (N, v) is *positive* if $v(S) > 0$ for all nonempty $S \subseteq N$.)

Let us mention that balanced values need not be core allocations nor the other way around, as is demonstrated by the following example.

Example 3. Consider the game

$$
v(S) = \begin{cases}
7 & \text{if } S = N, \\
5 & \text{if } S \in \{\{1, 3\}, \{2, 3\}\}, \\
4 & \text{if } S = \{1, 2\}, \\
2 & \text{if } S = \{3\}, \\
0 & \text{if } S \in \{\{1\}, \{2\}\}.
\end{cases}
$$

It can be verified that $(0, 28/9, 35/9)$, $(28/9, 0, 35/9)$, and $(7/4, 7/4, 7/2)$ are the balanced values for (N, v), while the core consists of the single point $(2, 2, 3)$.

However, for any monotone *simple* game (N, v), every core allocation is a balanced value. Recall that a game (N, v) is simple if $v(S) \in \{0, 1\}$ for all $S \subseteq N$.

Proposition 2. *If (N, v) is a simple monotone game, then $\mathrm{Core}(N, v) \subseteq B(N, v)$.*

Observe also that for every $(N, v) \in \mathcal{G}$ with $\Delta_{N,v}(S) \geq 0$ for all $S \subseteq N$, we have that $B(N, v) \subseteq$ Core(N, v). Indeed, if x^* is a balanced value of (N, v), then for every $T \subseteq N$ we have

$$x_T^* = \sum_{i \in T} h_i(x^*) = \sum_{i \in T} \left(\sum_{\substack{S \subseteq N, i \in S \\ x_S \neq 0}} \frac{x_i}{x_S} \Delta_{N,v}(S) + \sum_{\substack{S \subseteq N, i \in S \\ x_S = 0}} \frac{1}{|S|} \Delta_{N,v}(S) \right)$$

$$\geq \sum_{i \in T} \left(\sum_{\substack{S \subseteq T, i \in S \\ x_S \neq 0}} \frac{x_i}{x_S} \Delta_{N,v}(S) + \sum_{\substack{S \subseteq T, i \in S \\ x_S = 0}} \frac{1}{|S|} \Delta_{N,v}(S) \right)$$

$$= \sum_{\substack{S \subseteq T \\ x_S \neq 0}} \Delta_{N,v}(S) + \sum_{\substack{S \subseteq T \\ x_S = 0}} \Delta_{N,v}(S) = \sum_{S \subseteq T} \Delta_{N,v}(S) = v(T).$$

Note that many economic applications are modeled by games with nonnegative dividends such as river games (see Ambec and Sprumont, 2002), sequencing games (see Curiel et al., 1989), auction games (see Graham et al., 1990), dual airport games (see Littlechild and Owen, 1973), telecommunication games (see van den Nouweland et al., 1996), polluted river games (see Ni and Wang, 2007) and queuing games (see Maniquet, 2003).

4. Balanced transformation

In this section we introduce the notion of balanced transformation which we use to get an axiomatization of the balanced solution on monotone games.

Definition 4. Let N be a nonempty finite set. A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called *balanced* if there exists a nonnegative vector $\chi^{\mathcal{B}} \in \mathbb{R}^{2^N}$ with components $(\chi^{\mathcal{B}}(S))_{S \subseteq N}$ such that $\sum_{S \in \mathcal{B}, i \in S} \chi^{\mathcal{B}}(S) = 1$ for every $i \in N$, and $\chi^{\mathcal{B}}(S) = 0$ whenever $S \notin \mathcal{B}$. Such a $\chi^{\mathcal{B}}$ is called a *characteristic function* of $\mathcal{B}$.

Definition 5. Let $(N, v) \in \mathcal{G}$. We say that a game (N, w) is an *elementary balanced transformation of (N, v) with respect to $x \in \mathbb{R}^N$* if there exist $\alpha \in \mathbb{R}$ and a balanced collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ with a characteristic vector $\chi^{\mathcal{B}}$ such that

$$\Delta_{N,w}(S) = \Delta_{N,v}(S) + \alpha x_S \chi^{\mathcal{B}}(S)$$

for every $S \subseteq N$.

We say that a game (N, w) is a *balanced transformation of (N, v) with respect to $x \in \mathbb{R}^N$* if there exist games (N, v_0), ..., (N, v_k) such that

- $(N, v_0) = (N, v)$,
- $(N, v_k) = (N, w)$,

- (N, v_{j+1}) is an elementary balanced transformation of (N, v_j) with respect to x, $j = 0, \ldots, k-1$.

Remark 3. Observe that if (N, w) is a balanced transformation of (N, v), then also (N, v) is a balanced transformation of (N, w).

Definition 6. Let $\mathcal{C} \subseteq \mathcal{G}$ and F be a solution defined on $\mathcal{C}$. We say that F satisfies *balanced consistency* on $\mathcal{C}$ if, for every $(N, v) \in \mathcal{C}$, $x \in F(N, v)$, and every balanced transformation $(N, w) \in \mathcal{C}$ of (N, v) with respect to x, we have $\beta x \in F(N, w)$ for some $\beta \in \mathbb{R}$.

Each elementary balanced transformation of a game (N, v) with respect to a vector x adds an α-multiple of $x_S \chi^{\mathcal{B}}(S)$ to each dividend $\Delta_{N,v}(S)$ where S is chosen from some balanced family $\mathcal{B}$ and $\chi^{\mathcal{B}}$ is a characteristic vector of $\mathcal{B}$. Since the collection $\mathcal{B}$ is balanced, each player i "appears" in $\mathcal{B}$ (in sum) exactly "once". (In fact we can restrict ourselves to such $\mathcal{B}$ that are partitions of the player set N. Thus for each player i there exists exactly one coalition $S \in \mathcal{B}$ with $i \in S$.) Thus an elementary balanced transformation of the game (N, v) adds to (N, v) a multiple of a "basic" game $(N, v_x^{\mathcal{B}})$ which is defined by

$$\Delta_{N, v_x^{\mathcal{B}}}(S) = x_S \chi^{\mathcal{B}}(S).$$

A balanced transformation is done just by a finite repetition of the above modification where x is fixed, but balanced collections $\mathcal{B}$ and scale α can vary.

A *balanced consistent solution* F preserves ratios among components of the payoff vector whenever the game is a balanced transformation with respect to x from $F(N, v)$. This property with an "efficiency axiom" gives an axiomatization of the balanced solution for the class $\mathcal{G}_M$ (the class of all nonnegative monotone games). See the next two propositions.

Let us also note that our approach is similar to Shapley's in a sense. In his axiomatization, Shapley (1953b) employs *unanimity* games instead of $(N, v_x^{\mathcal{B}})$ and *additivity* instead of *balanced consistency*.

Proposition 3. *The balanced solution satisfies*

(i) *balanced consistency on $\mathcal{G}_M$,*
(ii) *component efficiency on $\mathcal{G}$.*

Proposition 4. *Let F be a nonnegative solution defined on $\mathcal{G}_M$ satisfying balanced consistency and component efficiency on $\mathcal{G}_M$. Then $F(N, v) = B(N, v)$ for every $(N, v) \in \mathcal{G}_M$.*

Since nonnegativity, component efficiency and balanced consistency are logically independent, they axiomatize the balanced value on $\mathcal{G}_M$.

Note that for positive two-player games component efficiency and balanced consistency imply proportional standardness for two-player games (Ortmann, 2000). Or, in other words, it establishes the equity principle of Homans (1961) and Selten (1978) for all positive two-player games, i.e., the solution satisfying component efficiency and balanced consistency redistributes the worth $v(N)$ proportionally to the singleton worths for all positive two-person games.

For general games (with more than two players), the proportionality principle is not obvious. The balanced consistency gives us the intuition what is the standard of comparison of the equity theory of Homans (1961) and Selten (1978). It is clear what the equity principle states for *inessential* games with non-zero sum of singleton worths, that is games (N, v) such that $v(S) = \sum_{i \in S} v(\{i\})$ for every $S \subseteq N$, for which $v_N = \sum_{i \in N} v(\{i\}) \neq 0$, namely that $v(N)$ is allocated proportional to the singleton worths. (In the equity theory the singleton worths cannot be employed as the key numbers in the case that they sum to zero.)[1]

Since a general monotone game has a balanced value, it is a balanced transformation of an inessential game. The key numbers of the general monotone game are therefore the singleton worths of the corresponding inessential game.

5. PROOFS

5.1. Proof of Theorem 1.

Some of the results presented in this subsection are standard (e.g., Lemma 1, 4 and 5). We include them for the convenience of the reader.

Let $(N, v) \in \mathcal{G}$. The symbol $X_0(N, v)$ denotes the set of all $x \in X(N, v)$ with nonnegative coordinates. Usually we will write just X_0 instead of $X_0(N, v)$.

Let $S \subseteq N$ and $x \in \mathbb{R}^N$ be a nonnegative vector (i.e., all coordinates of x are nonnegative) with $x_S > 0$. We set

$$k_S(x) = \sum_{\substack{T \subseteq N \\ T \supseteq S}} \frac{(-1)^{|T|-|S|}}{x_T}, \qquad q_S(x) = \sum_{\substack{T \subseteq N \\ T \supseteq S}} \frac{(-1)^{|T|-|S|}}{x_T^2}.$$

We establish properties of h_i, k_S, and q_S needed in the sequel. In the proofs we will abbreviate notation $\Delta_{N,v}(T)$ to $\Delta(T)$ in case there is no confusion about the game we consider.

---

[1]Note that for inessential games the equity principle is established by all the solutions that satisfy component efficiency, thus also by the Shapley value and the balanced solution.

Lemma 1. *Let $(N, v) \in \mathcal{G}$ and $i \in N$. Then*

$$h_i(x) = x_i \sum_{\substack{S \subseteq N \\ i \in S}} k_S(x)(v(S) - v(S \setminus \{i\}))$$

for $x \in X_0$ with $x_i > 0$.

Proof. It is known that the dividends can be expressed as

$$\Delta(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} v(S), \qquad T \subseteq N.$$

We can write

$$h_i(x) = \sum_{\substack{T \subseteq N \\ i \in T}} \frac{x_i}{x_T} \Delta(T) = \sum_{\substack{T \subseteq N \\ i \in T}} \frac{x_i}{x_T} \left(\sum_{S \subseteq T} (-1)^{|T| - |S|} v(S) \right)$$

$$= x_i \sum_{S \subseteq N} \sum_{\substack{T \subseteq N \\ T \supseteq S, i \in T}} (-1)^{|T| - |S|} \frac{1}{x_T} v(S)$$

$$= x_i \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{T \subseteq N \\ T \supseteq S, i \in T}} (-1)^{|T| - |S|} \frac{1}{x_T} v(S) + x_i \sum_{\substack{S \subseteq N \\ i \notin S}} \sum_{\substack{T \subseteq N \\ T \supseteq S, i \in T}} (-1)^{|T| - |S|} \frac{1}{x_T} v(S)$$

$$= x_i \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{T \subseteq N \\ T \supseteq S}} (-1)^{|T| - |S|} \frac{1}{x_T} v(S)$$

$$+ x_i \sum_{\substack{R \subseteq N \\ i \in R}} \sum_{\substack{T \subseteq N \\ T \supseteq R}} (-1)^{|T| - (|R| - 1)} \frac{1}{x_T} v(R \setminus \{i\})$$

$$= x_i \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{T \subseteq N \\ T \supseteq S}} (-1)^{|T| - |S|} \frac{1}{x_T} v(S) - x_i \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{T \subseteq N \\ T \supseteq S}} (-1)^{|T| - |S|} \frac{1}{x_T} v(S \setminus \{i\})$$

$$= x_i \sum_{\substack{S \subseteq N \\ i \in S}} k_S(x) \big(v(S) - v(S \setminus \{i\}) \big).$$

$\square$

Lemma 2. *Let N be a finite nonempty set, $S \subseteq N$, and $x \in \mathbb{R}^N$ be a nonnegative vector with $x_S > 0$. Then we have $k_S(x) > 0$ and $q_S(x) > 0$.*

Proof. We start with the following claim.

Claim 1. *Let g be a positive $\mathcal{C}^\infty$ function on $(0, +\infty)$ such that $(-1)^p g^{(p)}$ is positive on $(0, +\infty)$ for every $p \in \mathbb{N}$. Then for every nonempty finite set N, $S \subseteq N$, and every nonnegative vector $x \in \mathbb{R}^N$*

with $x_S > 0$ we have

$$\sum_{\substack{T \subseteq N \\ T \supseteq S}} (-1)^{|T|-|S|} g(x_T) > 0.$$

Proof. Fix a nonempty finite set N and denote $|N| = n$. If $|S| = |N|$, then the assertion obviously holds since g is positive. Assume that the assertion is valid for this N and every g, S, and x satisfying the required properties and moreover $|S| > k$, where $k < |N|$.

We are going to prove the assertion for $S \subseteq N$ with $|S| = k$ and nonnegative $x \in \mathbb{R}^N$ with $x_S > 0$. Take $i \in N \setminus S$. Define an auxiliary function ψ by

$$\psi(z) = \sum_{\substack{T \subseteq N \\ T \supseteq \bar{S} \cup \{i\}}} (-1)^{|T|-|S|} g(x_{T \setminus \{i\}} + z) + \sum_{\substack{T \subseteq N \\ T \supseteq \bar{S}, i \notin T}} (-1)^{|T|-|S|} g(x_T), \qquad z \in [0, \infty).$$

We have

$$\psi(0) = \sum_{\substack{L \subseteq N \\ L \supseteq \bar{S}, i \notin L}} -(-1)^{|L|-|S|} g(x_L) + \sum_{\substack{T \subseteq N \\ T \supseteq \bar{S}, i \notin T}} (-1)^{|T|-|S|} g(x_T) = 0. \tag{2}$$

We compute the first derivative of ψ

$$\psi'(z) = \sum_{\substack{T \subseteq N \\ T \supseteq \bar{S} \cup \{i\}}} (-1)^{|T|-|S|} g'(x_{T \setminus \{i\}} + z).$$

Set $P := S \cup \{i\}$. Using the induction hypothesis, we obtain

$$\psi'(z) = \sum_{\substack{T \subseteq N \\ T \supseteq P}} (-1)^{|T|-|P|} (-g')(x_{T \setminus \{i\}} + z) > 0 \qquad \text{for } z \in [0, +\infty). \tag{3}$$

Here we have used the fact that the function $-g' = (-1)^1 g^{(1)}$ is positive and satisfies the required conditions on signs of its derivatives. Using (2) and (3), we obtain

$$\psi(x_i) = \sum_{\substack{T \subseteq N \\ T \supseteq S}} (-1)^{|T|-|S|} g(x_T) > 0$$

and the claim is proved. $\qquad \square$

Applying Claim 1 to the functions $g(t) = 1/t$ and $g(t) = 1/t^2$ the proof of Lemma 2 is finished. $\quad \square$

Throughout this subsection and in Subsections 5.2 and 5.3, we will assume that the considered game (N, v) is *monotone* and satisfies $v(N) > 0$. Then we have that the corresponding set X_+ is nonempty. If $v(N) = 0$, then the assertions of Theorem 1, Theorem 2, and Proposition 1 are trivial or vacuous. Again, we denote $|N|$ by n.

Lemma 3. (i) *The mapping h is continuous on X_+.*

(ii) *We have $h(x) \in X_0$ for all $x \in X_+$.*

Proof. The statement (i) is obvious. As for (ii), recall that $h(x) \in X$ for every $x \in X$ by Remark 1. Using monotonicity of the game (N, v), Lemma 1 and Lemma 2, we see that $h_i(x) \geq 0$ for every $x \in X_+$ and $i \in N$. Thus we have $h(x) \in X_0$ for $x \in X_+$. $\qquad\square$

Now we define the mapping $H : X_0 \to 2^{\mathbb{R}^N}$ by

$$H(x) = \{\alpha \in \mathbb{R}^n \mid \text{there exists a sequence } (x^j)$$

$$\text{of points of } X_+ \text{ such that } x^j \to x \text{ and } h(x^j) \to \alpha\}.$$

Lemma 4.   (i) *The set* $\{(x, y) \in X_0 \times X_0 \mid y \in H(x)\}$ *is closed.*

   (ii) *The set* $H(x)$ *is a nonempty compact subset of* X_0 *for every* $x \in X_0$.

   (iii) *We have* $H(x) = \{h(x)\}$ *for every* $x \in X_+$.

Proof. (i) Take sequences (x^j), $x^j \in X_0$, and (y^j) such that $x^j \to x \in X_0$, $y^j \in H(x^j)$, and $y^j \to y$. For each $j \in \mathbb{N}$ there exists $z^j \in X_+$ such that $||z^j - x^j|| < 1/j$ and $||h(z^j) - y^j|| < 1/j$. Then $z^j \to x$ and $h(z^j) \to y$. Consequently, $y \in H(x)$.

(ii) Fix $x \in X_0$. Since X_0 is compact, we have $H(x) \subseteq X_0$ by Lemma 3(ii). Using (i) and compactness of X_0, we get that $H(x)$ is compact. To prove that $H(x) \neq \emptyset$ take a sequence (x^j), $x^j \in X_+$, with $x^j \to x$. By Lemma 3(ii) the sequence $(h(x^j))$ is contained in the compact set X_0. Therefore, there exists a convergent subsequence $(h(x^{j_k}))_{k=1}^{\infty}$ with a limit $\alpha \in X_0$. Thus $\alpha \in H(x)$, showing that $H(x) \neq \emptyset$.

(iii) This follows from continuity of h on X_+. $\qquad\square$

Now let us define a mapping F from X_0 to the set of all convex subsets of X_0 such that $F(x)$ is the convex envelope of $H(x)$ for every $x \in X_0$.

Lemma 5.   (i) *The set* $F(x)$ *is a nonempty convex compact subset of* X_0 *for all* $x \in X_0$.

   (ii) *We have* $F(x) = \{h(x)\}$ *for every* $x \in X_+$.

   (iii) *The set* $\{(x, y) \in X_0 \times X_0 \mid y \in F(x)\}$ *is closed.*

Proof. (i) This assertion immediately follows from Lemma 4(ii), convexity of X_0, and the well-known fact that the convex envelope of any compact subset of $\mathbb{R}^n$ is compact.

(ii) This part clearly follows from Lemma 4(iii).

(iii) Take sequences (x^j), $x^j \in X_0$, and (y^j) such that $x^j \to x \in X_0$, $y^j \in F(x^j)$, and $y^j \to y$. Now we use Carathéodory's theorem, which asserts that each element of the convex envelope of a set $M \subseteq \mathbb{R}^{n-1}$ can be written as a convex combination of n elements of the set M. Since the

simplex X_0 is $n-1$ dimensional, there are $\alpha_1^j, \ldots, \alpha_n^j \in [0,1]$ and $y^{j,1}, \ldots, y^{j,n} \in H(x^j)$ such that

$$\alpha_1^j y^{j,1} + \cdots + \alpha_n^j y^{j,n} = y^j \quad \text{and} \quad \sum_{s=1}^n \alpha_s^j = 1.$$

Going to subsequences, if necessary, we may assume that $\alpha_s^j \to \alpha_s \in [0,1]$, and $y^{j,s} \to y^{\infty,s}$. Then

$$\alpha_1 y^{\infty,1} + \cdots + \alpha_n y^{\infty,n} = y \quad \text{and} \quad \sum_{s=1}^n \alpha_s = 1.$$

Since the graph of H is closed by Lemma 4(i), we have that $y^{\infty,s} \in H(x)$, and thus $y \in F(x)$. $\quad\square$

Kakutani's theorem (see Kakutani, 1941, or, e.g., Franklin, 1980) states that any multivalued F from a nonempty compact convex subset D of $\mathbb{R}^N$ to itself such that the graph of F is closed and $F(x)$ is convex, closed, and nonempty for all $x \in D$, has a fixed point, i.e., there exists an $x^* \in D$ such that $x^* \in F(x^*)$.

Since we have shown that F and its domain X_0 satisfy the assumptions of Kakutani's theorem, we have the following lemma.

Lemma 6. *There exists $x^* \in X_0$ such that $x^* \in F(x^*)$.*

The next lemma shows the relationships between fixed points of h and F.

Lemma 7. *Let $x^* \in X_0$ be such that $x^* \in F(x^*)$. Then $h(x^*) = x^*$. Moreover, if we set $Q = \{i \in N \mid x_i^* = 0\}$, then $v(Q) = 0$.*

Proof. We distinguish two cases.

a) If $x^* \in X_+$ then $x^* \in F(x^*) = \{h(x^*)\}$ (Lemma 5(ii)) and, consequently, $x^* = h(x^*)$.

b) Suppose now that $x^* \in X_0 \backslash X_+$. From the definition of F there are elements $z^1, \ldots, z^p \in H(x^*)$ and $\beta_1, \ldots, \beta_p \in (0,1]$ such that $\sum_{s=1}^p \beta_s = 1$ and

$$\beta_1 z^1 + \cdots + \beta_p z^p = x^*. \tag{4}$$

Denote $Q = \{i \in N \mid x_i^* = 0\}$. Since $z_i^j \geq 0$, $j = 1, \ldots, p$, the equation (4) guarantees that $z_i^j = 0$ for every $i \in Q$, $j \in \{1, \ldots, p\}$. Let us simplify the notation by setting $z := z^1$. Since $z \in H(x^*)$ there exists a sequence (x^j), $x^j \in X_+$, such that $x^j \to x^*$ and $h(x^j) \to z$.

For $i \in Q$ we have $\lim_{j \to \infty} x_i^j = 0$ and $\lim_{j \to \infty} h_i(x^j) = 0$. Further, we get

$$\sum_{i \in Q} h_i(x^j) = \sum_{i \in Q} \left(\sum_{S \subseteq N, i \in S} \frac{x_i^j}{x_S^j} \Delta(S) \right)$$

$$= \sum_{i \in Q} \left(\left(\sum_{\substack{S \subseteq N, \\ i \in S, S \setminus Q \neq \emptyset}} \frac{x_i^j}{x_S^j} \Delta(S) \right) + \left(\sum_{S \subseteq Q, i \in S} \frac{x_i^j}{x_S^j} \Delta(S) \right) \right)$$

$$= \sum_{i \in Q} \left(\sum_{\substack{S \subseteq N \\ i \in S, S \setminus Q \neq \emptyset}} \frac{x_i^j}{x_S^j} \Delta(S) \right) + \sum_{i \in Q} \left(\sum_{S \subseteq Q, i \in S} \frac{x_i^j}{x_S^j} \Delta(S) \right)$$

$$= \sum_{i \in Q} \left(\sum_{\substack{S \subseteq N \\ i \in S, S \setminus Q \neq \emptyset}} \frac{x_i^j}{x_S^j} \Delta(S) \right) + \left(\sum_{S \subseteq Q} \Delta(S) \right)$$

$$= \sum_{i \in Q} \left(\sum_{\substack{S \subseteq N \\ i \in S, S \setminus Q \neq \emptyset}} \frac{x_i^j}{x_S^j} \Delta(S) \right) + v(Q).$$

The limit of the left side is

$$\lim_{j \to \infty} \sum_{i \in Q} h_i(x^j) = 0,$$

and, since $S \setminus Q \neq \emptyset$ implies $x_S^* > 0$, we have also

$$\lim_{j \to \infty} \sum_{i \in Q} \left(\sum_{\substack{S \subseteq N \\ i \in S, S \setminus Q \neq \emptyset}} \frac{x_i^j}{x_S^j} \Delta(S) \right) = 0.$$

Therefore, we have $v(Q) = 0$. This proves the second part of our statement.

By monotonicity of (N, v) we get $v(S) = 0$ for every $S \subseteq Q$. Consequently, $\Delta(S) = 0$ for all $S \subseteq Q$. Then

$$h_i(x) = \sum_{\substack{S \subseteq N \\ i \in S, x_S \neq 0}} \frac{x_i}{x_S} \Delta(S) + \sum_{\substack{S \subseteq N \\ i \in S, x_S = 0}} \frac{1}{|S|} \Delta(S)$$

$$= \sum_{\substack{S \subseteq N, i \in S \\ S \setminus Q \neq \emptyset, x_S \neq 0}} \frac{x_i}{x_S} \Delta(S) + \sum_{\substack{S \subseteq N, i \in S \\ S \setminus Q \neq \emptyset, x_S = 0}} \frac{1}{|S|} \Delta(S), \qquad x \in X_0.$$

Since $x_i^* > 0$ for all $i \in N \setminus Q$, there exists a neighborhood V of x^* such that, for every $x \in X_0 \cap V$ and $S \subseteq N$ with $S \setminus Q \neq \emptyset$, we have $x_S > 0$ and

$$h_i(x) = \sum_{\substack{S \subseteq N, i \in S \\ S \setminus Q \neq \emptyset, x_S \neq 0}} \frac{x_i}{x_S} \Delta(S).$$

From this we conclude that h is continuous at x^*.

From the continuity of h at fixed point x^* of F it follows that $x^* \in F(x^*) = \{h(x^*)\}$, and thus $x^* = h(x^*)$. $\square$

Now we immediately see that the assertion of Theorem 1 holds.

5.2. Proof of Theorem 2.

Lemma 8. *For every $x \in X_0$ we have $h(x) \in F(x)$.*

Proof. Let $x \in X_0$. Denote $Q = \{i \in N \mid x_i = 0\}$ and for $\varepsilon > 0$ we set

$$
y_i^\varepsilon =
\begin{cases}
\varepsilon & \text{for } i \in Q; \\
x_i - \dfrac{|Q|}{|N| - |Q|}\varepsilon & \text{for } i \in N \setminus Q.
\end{cases}
$$

If $\varepsilon > 0$ is sufficiently small, then $y^\varepsilon \in X_+$ and we have

$$
h_i(y^\varepsilon) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{y_i^\varepsilon}{y_S^\varepsilon} \Delta(S) = \sum_{\substack{S \subseteq N \\ i \in S, x_S \neq 0}} \frac{y_i^\varepsilon}{y_S^\varepsilon} \Delta(S) + \sum_{\substack{S \subseteq N \\ i \in S, x_S = 0}} \frac{\varepsilon}{|S|\varepsilon} \Delta(S), \qquad i \in Q.
$$

Now it is easy to see that $h(y^\varepsilon) \to h(x)$ for $\varepsilon \to 0+$. This shows $h(x) \in H(x) \subseteq F(x)$ and we are done. $\square$

Now we prove Theorem 2. By Theorem 1, the set $B(N, v)$ is nonempty. Using Lemma 1 and superadditivity of (N, v), we get for $x \in X_0$ with $x_i > 0$ the following estimates.

$$
\begin{aligned}
h_i(x) &= x_i \sum_{\substack{S \subseteq N \\ i \in S}} k_S(x)\, (v(S) - v(S \setminus \{i\})) \\
&\geq x_i \sum_{\substack{S \subseteq N \\ i \in S}} (k_S(x) \cdot v(\{i\})) = x_i \cdot v(\{i\}) \cdot \sum_{\substack{S \subseteq N \\ i \in S}} k_S(x).
\end{aligned}
\tag{5}
$$

The term $\sum_{\substack{S \subseteq N \\ i \in S}} k_S(x)$ can be rewritten as follows

$$
\begin{aligned}
\sum_{\substack{S \subseteq N \\ i \in S}} k_S(x) &= \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{T \subseteq N \\ T \supseteq S}} \frac{(-1)^{|T| - |S|}}{x_T} \\
&= \sum_{\substack{T \subseteq N \\ i \in T}} \sum_{\substack{S \subseteq T \\ i \in S}} \frac{(-1)^{|T| - |S|}}{x_T} \\
&= \sum_{\substack{T \subseteq N \\ i \in T}} \left(\frac{(-1)^{|T|}}{x_T} \cdot \sum_{S \subseteq T, i \in S} (-1)^{-|S|} \right).
\end{aligned}
$$

Further, we compute

$$\sum_{S \subseteq T, i \in S} (-1)^{-|S|} = \sum_{R \subseteq T \setminus \{i\}} (-1)^{-|R|-1} = -\sum_{n=0}^{|T|-1} \sum_{\substack{R \subseteq T \setminus \{i\} \\ |R|=n}} (-1)^{-n}$$

$$= -\left(\binom{|T|-1}{0} - \binom{|T|-1}{1} + \cdots + (-1)^{|T|-1} \binom{|T|-1}{|T|-1} \right)$$

$$= \begin{cases} -1 & \text{for } |T| = 1, \\ -(1-1)^{|T|-1} = 0 & \text{for } |T| > 1. \end{cases}$$

Thus we get

$$\sum_{\substack{S \subseteq N \\ i \in S}} k_S(x) = \sum_{\substack{T \subseteq N \\ i \in T, |T|=1}} \frac{(-1)^{|T|+1}}{x_T} = \frac{1}{x_i}. \tag{6}$$

The inequality (5) and the identity (6) yield $h_i(x) \geq v(\{i\})$ for $x \in X_0$ with $x_i > 0$.

Now let x^* be a balanced value of the game (N, v). Denote $Q = \{i \in N \mid x_i^* = 0\}$. If $i \in N \setminus Q$, then we have $x_i^* > 0$ and so $x_i^* = h_i(x^*) \geq v(\{i\})$ as shown above.

By Lemmas 7 and 8 we have $v(Q) = 0$. By monotonicity this leads to $v(\{i\}) = 0$ for all $i \in Q$, and we have $x_i^* = v(\{i\})$ for every $i \in Q$, showing individual rationality of the balanced solution for monotone superadditive games.

5.3. **Proof of Proposition 1.** We again use the same notation as in the previous subsections.

(i) Set $Z = \{x \in X_0 \mid x_i = x_j\}$. The set Z is nonempty, compact, and convex. Let F be the mapping defined in Subsection 5.1 after Lemma 4. We define a multivalued mapping G by $G(x) = F(x) \cap Z$. Using the symmetry of the players i and j, we have that $h_i(x) = h_j(x)$ whenever $x \in Z$. This and Lemma 8 implies that $h(x) \in G(x)$ for every $x \in Z$. Consequently, $G(x) \neq \emptyset$ for every $x \in Z$. Further, it is clear that $G(x)$ is a compact convex set and the set $\{(x,y) \in Z \times Z \mid y \in G(x)\}$ is closed (see Lemma 5(iii)). Applying Kakutani's theorem, we get a fixed point $x^* \in Z$ of the mapping G. Then clearly x^* is a fixed point of F and $x_i^* = x_j^*$. By Lemma 7 we get that x^* is a fixed point of h.

(ii) The idea of this proof is the same as in the previous part. Set $R = \{x \in X_0 \mid x_i = 0\}$. The set R is nonempty, compact, and convex. Let F again be the mapping defined in Subsection 5.1. We define a multivalued mapping G by $G(x) = F(x) \cap R$, $x \in R$. It is clear that $G(x)$ is a compact convex set for every $x \in R$ and the set $\{(x,y) \in R \times R \mid y \in G(x)\}$ is closed.

We also show that $G(x)$ is nonempty whenever $x \in R$. To this end, fix $x \in R$. Since $v(N) > 0$, one can find $j \in N$ with $x_j > 0$. We define $y^\varepsilon \in X$ by

$$
y_k^\varepsilon = \begin{cases}
\varepsilon^2 & \text{for } k = i; \\
x_k + \varepsilon & \text{for } k \in N \setminus \{i, j\}; \\
x_j - \varepsilon^2 - (|N| - 2)\varepsilon & \text{for } k = j.
\end{cases}
$$

For every $\varepsilon \geq 0$ we have $y^\varepsilon \in X$ and for sufficiently small $\varepsilon > 0$ we have $y^\varepsilon \in X_+$. Then we have $\lim_{\varepsilon \to 0+} y^\varepsilon = x$, and a straightforward computation results in $\lim_{\varepsilon \to 0+} y_i^\varepsilon / y_S^\varepsilon = 0$ for every nonempty $S \subseteq N$ with $S \neq \{i\}$. Using this and $\Delta(\{i\}) = 0$, we infer $\lim_{\varepsilon \to 0+} h_i(y^\varepsilon) = 0$. This implies that there exists a sequence (y^j) of elements of X_+ going to x such that $(h(y^j))_j$ converges to some $\alpha \in X_0$ with $\alpha_i = 0$. Thus $\alpha \in F(x) \cap R$ and $G(x) \neq \emptyset$.

Applying Kakutani's theorem, we obtain a fixed point $x^* \in R$ of the mapping G. Thus $x_i^* = 0$ and $x^* \in F(x^*)$. By Lemma 7 we get that x^* is a fixed point of h.

(iii) Suppose that $v(\{i\}) > 0$ and x^* is a balanced value of (N, v). Applying Lemma 8, we get $x^* \in F(x^*)$. Now from Lemma 7 it follows that $x_i^* > 0$ since otherwise $v(\{i\}) = 0$.

(iv) Without any loss of generality we may assume that $k = 1$, $l = 2$. Let x^* be a balanced value and assume to the contrary that $x_1^* < x_2^*$. By Proposition 1(iii) we have $x_1^* > 0$. By assumption we have

$$
\begin{aligned}
v(T \cup \{1\}) - v(T) &\geq v(T \cup \{2\}) - v(T), \text{ and} \\
v(T \cup \{1, 2\}) - v(T \cup \{2\}) &\geq v(T \cup \{1, 2\}) - v(T \cup \{1\})
\end{aligned}
\tag{7}
$$

for every $T \subseteq N \setminus \{1, 2\}$.

Fix $T \subseteq N \setminus \{1, 2\}$ and define an auxiliary function m by

$$
m(z) = \sum_{T \subseteq R \subseteq N \setminus \{1,2\}} (-1)^{|R| - |T|} \frac{1}{x_R^* + z}
$$

for $z > 0$. If $N = \{1, 2\}$ then m is obviously decreasing. If $N \setminus \{1, 2\} \neq \emptyset$, then the function m is decreasing since

$$
m'(z) = -\sum_{T \subseteq R \subseteq N \setminus \{1,2\}} (-1)^{|R| - |T|} \frac{1}{(x_R^* + z)^2} < 0
$$

by Lemma 2. This gives

$$
\begin{aligned}
k_{T \cup \{1\}}(x^*) - k_{T \cup \{2\}}(x^*) &= \sum_{T \subseteq R \subseteq N \setminus \{1,2\}} (-1)^{|R| - |T|} \frac{1}{x_R^* + x_1^*} \\
&\quad - \sum_{T \subseteq R \subseteq N \setminus \{1,2\}} (-1)^{|R| - |T|} \frac{1}{x_R^* + x_2^*} > 0.
\end{aligned}
\tag{8}
$$

Now we write

$$g_1(x^*) := \sum_{S \subseteq N \setminus \{1,2\}} k_{S \cup \{1\}}(x^*)(v(S \cup \{1\}) - v(S))$$

$$+ \sum_{S \subseteq N \setminus \{1,2\}} k_{S \cup \{1,2\}}(x^*)(v(S \cup \{1,2\}) - v(S \cup \{2\})),$$

$$g_2(x^*) := \sum_{S \subseteq N \setminus \{1,2\}} k_{S \cup \{2\}}(x^*)(v(S \cup \{2\}) - v(S))$$

$$+ \sum_{S \subseteq N \setminus \{1,2\}} k_{S \cup \{1,2\}}(x^*)(v(S \cup \{1,2\}) - v(S \cup \{1\})).$$

Using (7) a (8) we compare the above sums. This yields $g_1(x^*) > g_2(x^*)$. On the other hand, since x_1^* and x_2^* are positive we have $x_1^* = h_1(x^*) = x_1^* g_1(x^*)$ and $x_2^* = h_2(x^*) = x_2^* g_2(x^*)$. Thus we get $g_1(x^*) = g_2(x^*) = 1$, yielding a contradiction.

(v) The balanced solution satisfies component efficiency by Proposition 3(ii). Since $C := \{i\}$, where i is a null player in (N, v), is a component in (N, v) we immediately get the conclusion.

5.4. **Proof of Proposition 2.** Let (N, v) be a simple monotone game. Suppose that $x \in \text{Core}(N, v)$. If $\Delta_{N,v}(S) \neq 0$, then $x_S = 1$. Indeed, $\Delta_{N,v}(S) \neq 0$ implies $v(S) \neq 0$. This yields $1 \geq x_S \geq v(S) = 1$, showing $x_S = 1$. Applying this observation, we get

$$h_i(x) = \sum_{\substack{S \subseteq N, i \in S \\ x_S = 1}} \frac{x_i}{x_S} \Delta_{N,v}(S) = x_i \sum_{\substack{S \subseteq N, i \in S \\ x_S = 1}} \Delta_{N,v}(S) = x_i \sum_{S \subseteq N, i \in S} \Delta_{N,v}(S), \qquad i \in N.$$

The last equality holds since we added just the zero dividends. Since the last sum equals $v(N) = 1$, we have $h_i(x) = x_i$ and we are done.

5.5. **Proof of Proposition 3.** (i) Suppose that x is a balanced value of $(N, v) \in \mathcal{G}_M$. This means that x is a nonnegative vector with

$$x_i = \sum_{\substack{S \subseteq N, i \in S \\ x_S \neq 0}} \frac{x_i}{x_S} \Delta_{N,v}(S) + \sum_{\substack{S \subseteq N, i \in S \\ x_S = 0}} \frac{1}{|S|} \Delta_{N,v}(S)$$

for every $i \in N$. Let $(N, w) \in \mathcal{G}_M$ be a balanced transformation of (N, v) w.r.t. x. Thus

$$\Delta_{N,w}(S) = \Delta_{N,v}(S) + \sum_{j=1}^{k} \alpha_j x_S \chi^{\mathcal{B}_j}(S)$$

for every $S \subseteq N$, where $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are balanced families with the corresponding characteristic vectors $\chi^{\mathcal{B}_1}, \ldots, \chi^{\mathcal{B}_k}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Denote $\alpha = \sum_{j=1}^{k} \alpha_j$. We distinguish two cases.

1) We assume $\alpha \neq -1$ and we set $y = (1+\alpha)x$. Then we have

$$
\begin{aligned}
h_i(N,w)(y) &= \sum_{\substack{S\subseteq N, i\in S \\ y_S\neq 0}} \frac{y_i}{y_S} \Delta_{N,w}(S) + \sum_{\substack{S\subseteq N, i\in S \\ y_S=0}} \frac{1}{|S|} \Delta_{N,w}(S) \\
&= \sum_{\substack{S\subseteq N, i\in S \\ x_S\neq 0}} \frac{(1+\alpha)x_i}{(1+\alpha)x_S} \Delta_{N,w}(S) + \sum_{\substack{S\subseteq N, i\in S \\ x_S=0}} \frac{1}{|S|} \Delta_{N,w}(S) \\
&= \sum_{\substack{S\subseteq N, i\in S \\ x_S\neq 0}} \frac{x_i}{x_S} \left(\Delta_{N,v}(S) + \sum_{j=1}^{k} \alpha_j x_S \chi^{\mathcal{B}_j}(S) \right) + \sum_{\substack{S\subseteq N, i\in S \\ x_S=0}} \frac{1}{|S|} \left(\Delta_{N,v}(S) + \sum_{j=1}^{k} \alpha_j x_S \chi^{\mathcal{B}_j}(S) \right) \\
&= \sum_{\substack{S\subseteq N, i\in S \\ x_S\neq 0}} \frac{x_i}{x_S} \Delta_{N,v}(S) + \sum_{\substack{S\subseteq N, i\in S \\ x_S\neq 0}} \frac{x_i}{x_S} \left(\sum_{j=1}^{k} \alpha_j x_S \chi^{\mathcal{B}_j}(S) \right) + \sum_{\substack{S\subseteq N, i\in S \\ x_S=0}} \frac{1}{|S|} \Delta_{N,v}(S) \\
&= x_i + \sum_{\substack{S\subseteq N, i\in S \\ x_S\neq 0}} x_i \left(\sum_{j=1}^{k} \alpha_j \chi^{\mathcal{B}_j}(S) \right) \\
&= x_i + \sum_{S\subseteq N, i\in S} x_i \left(\sum_{j=1}^{k} \alpha_j \chi^{\mathcal{B}_j}(S) \right) \qquad \text{(we added at most some zero terms)} \\
&= x_i \left(1 + \sum_{j=1}^{k} \alpha_j \sum_{S\subseteq N, i\in S} \chi^{\mathcal{B}_j}(S) \right) = (1+\alpha)x_i = y_i.
\end{aligned}
$$

From the above equalities it also follows that $(1+\alpha)x_N = y_N = w(N) \geq 0$. Then $1+\alpha \geq 0$ or $x_N = 0$. In both cases we get that y is a nonnegative vector. Consequently, we have that $y = (1+\alpha)x$ is a balanced value of (N,w).

2) If $\alpha = -1$, then we have

$$
\begin{aligned}
w(N) &= \sum_{S\subseteq N} \Delta_{N,w}(S) = \sum_{S\subseteq N} \left(\Delta_{N,v}(S) + \sum_{j=1}^{k} \alpha_j x_S \chi^{\mathcal{B}_j}(S) \right) \\
&= v(N) + \sum_{S\subseteq N} \sum_{j=1}^{k} \alpha_j x_S \chi^{\mathcal{B}_j}(S) = v(N) + \sum_{j=1}^{k} \left(\alpha_j \sum_{S\subseteq N} \left(\sum_{i\in S} x_i \cdot \chi^{\mathcal{B}_j}(S) \right) \right) \\
&= v(N) + \sum_{j=1}^{k} \alpha_j x_N = \left(1 + \sum_{j=1}^{k} \alpha_j \right) v(N) = (1+\alpha)v(N) = 0.
\end{aligned}
$$

Since $(N,w) \in \mathcal{G}_M$ we get $w(S) = 0$ for every $S \subseteq N$. Then the zero vector y is clearly a balanced value of (N,w) and we are done since $y = 0 \cdot x$.

(ii) We start with the following claim.

Claim 2. *Let $(N, v) \in \mathcal{G}$, C be a component of (N, v), and $i \in C$. Then*

$$h_i(N, v)(y) = h_i(C, v_C)(y|_C)$$

for every $y \in X(N, v)$, where (C, v_C) is the restriction of (N, v) to C, i.e., $v_C(S) = v(S)$ for every $S \subseteq C$.

Proof. First of all, we prove that if $S \subseteq N$ intersects both C and $N \setminus C$, then $\Delta_{N,v}(S) = 0$. If S contains just two elements, then the observation follows directly from the definition of component. Now take a set S, $|S| > 2$, with the required property and assume that the observation holds for all sets having less elements than S. Using this assumption and the definition of component we have

$$\Delta_{N,v}(S) = v(S) - \sum_{T \subsetneq S} \Delta_{N,v}(T) = v(S) - \sum_{T \subseteq S \cap C} \Delta_{N,v}(T) - \sum_{T \subseteq S \setminus C} \Delta_{N,v}(T)$$

$$= v(S) - v(S \cap C) - v(S \setminus C) = 0.$$

Then the assertion of Claim 2 follows immediately from the observation that $\Delta_{N,v}(S) = \Delta_{C,v_C}(S)$ whenever $S \subseteq C$. $\qquad\square$

To prove component efficiency consider $(N, v) \in \mathcal{G}$ with a component C and $x \in B(N, v)$. Using Claim 2 we get

$$x_C = \sum_{i \in C} h_i(N, v)(x) = \sum_{i \in C} h_i(C, v_C)(x|_C) = v(C).$$

The last equality follows from Remark 1.

5.6. **Proof of Proposition 4.** Let $(N, v) \in \mathcal{G}_M$. Fix a nonnegative vector $x \in \mathbb{R}^N$. To prove our proposition we define a special balanced transformation (N, w) of (N, v) with respect to x.

Set $Q = \{i \in N \mid x_i = 0\}$. For every $S \subseteq N$, $|S| \geq 2$, we define a balanced family $\mathcal{B}_S = \{S\} \cup \{\{i\} \mid i \in N \setminus S\}$. Further we define the balanced family $\mathcal{B}_0 = \{\{1\}, \{2\}, \ldots, \{n\}\}$. The corresponding characteristic vector $\chi^{\mathcal{B}_S}$ is defined by $\chi^{\mathcal{B}_S}(R) = 1$ for $R \in \mathcal{B}_S$ and 0 otherwise. The characteristic vector $\chi^{\mathcal{B}_0}$ is defined in the same way as $\chi^{\mathcal{B}_0}(R) = 1$ for $R \in \mathcal{B}_0$ and $\chi^{\mathcal{B}_0}(R) = 0$ otherwise.

We set

$$\alpha_S = -\frac{\Delta_{N,v}(S)}{x_S},$$

for every $S \subseteq N$ with $S \not\subseteq Q$, $|S| \geq 2$, and

$$\alpha_0 = \sum_{\substack{S \subseteq N \\ S \not\subseteq Q, |S| \geq 2}} \frac{\Delta_{N,v}(S)}{x_S}.$$

The dividends of (N, w), and thus the game w itself, are defined by

$$\Delta_{N,w}(S) = \Delta_{N,v}(S) + \alpha_0 x_S \chi^{\mathcal{B}_0}(S) + \sum_{\substack{T \subseteq N \\ T \not\subseteq Q, |T| \geq 2}} \alpha_T x_S \chi^{\mathcal{B}_T}(S).$$

Now, for every $S \subseteq N$ with $S \not\subseteq Q$, $|S| \geq 2$, we have

$$\Delta_{N,w}(S) = \Delta_{N,v}(S) - \frac{\Delta_{N,v}(S)}{x_S} x_S = 0,$$

showing that Q is a component of (N, w). Further, the equality $\Delta_{N,w}(S) = \Delta_{N,v}(S)$ for $S \subseteq Q$ follows immediately from the definition of (N, w). Thus, for every $S \subseteq N$, we have

$$w(S) = w(S \cap Q) + \sum_{i \in S \setminus Q} w(\{i\}) = v(S \cap Q) + \sum_{i \in S \setminus Q} w(\{i\}). \tag{9}$$

For $i \in N \setminus Q$ we have

$$w(\{i\}) = \Delta_{N,w}(\{i\}) = \Delta_{N,v}(\{i\}) + x_i \sum_{\substack{S \subseteq N \\ S \not\subseteq Q, |S| \geq 2}} \frac{\Delta_{N,v}(S)}{x_S} - x_i \sum_{\substack{T \subseteq N \\ T \not\subseteq Q, |T| \geq 2, i \notin T}} \frac{\Delta_{N,v}(T)}{x_T}$$

$$= v(\{i\}) + x_i \sum_{\substack{S \subseteq N, S \not\subseteq Q \\ |S| \geq 2, i \in S}} \frac{\Delta_{N,v}(S)}{x_S}$$

$$= v(\{i\}) + x_i \sum_{\substack{S \subseteq N \\ |S| \geq 2, i \in S}} \frac{\Delta_{N,v}(S)}{x_S} = x_i \sum_{S \subseteq N, i \in S} \frac{\Delta_{N,v}(S)}{x_S}$$

$$= h_i(N, v)(x) \geq 0 \qquad \text{(by Lemmas 1 and 2).}$$

Using (9) we infer that $(N, w) \in \mathcal{G}_M$.

Further we have $w(N) = v(N)$. Indeed, we can compute

$$w(N) = \sum_{S \subseteq N} \Delta_{N,w}(S) = \sum_{S \subseteq N} \Delta_{N,v}(S) + \sum_{S \subseteq N} \alpha_0 x_S \chi^{\mathcal{B}_0}(S) + \sum_{S \subseteq N} \sum_{\substack{T \subseteq N \\ T \not\subseteq Q, |T| \geq 2}} \alpha_T x_S \chi^{\mathcal{B}_T}(S)$$

$$= v(N) + \sum_{i \in N} \alpha_0 x_i + \sum_{\substack{T \subseteq N \\ T \not\subseteq Q, |T| \geq 2}} \sum_{S \subseteq N} \alpha_T x_S \chi^{\mathcal{B}_T}(S)$$

$$= v(N) + \alpha_0 x_N + \sum_{\substack{T \subseteq N \\ T \not\subseteq Q, |T| \geq 2}} \alpha_T x_N = v(N).$$

Now moreover assume that $x \in F(N, v)$. Since F satisfies balanced consistency, there exists $\beta \in \mathbb{R}$ such that $\beta x \in F(N, w)$. Component efficiency of F and the equality $w(N) = v(N)$ gives $\beta x_N = w(N) = v(N) = x_N$. Since F is a nonnegative solution, we see that $\beta = 1$ or x is the zero vector. In both cases we have $x \in F(N, w)$. From component efficiency of F we get $w(Q) = x_Q = 0$. Combining this fact with (9) we see that the game (N, w) is inessential, that is $w(S) = \sum_{i \in S} w(\{i\})$

for every $S \subseteq N$. Component efficiency of F then gives $x_i = w(\{i\})$ for every $i \in N$. Consequently, x is a balanced value of (N, w). The game (N, v) is a balanced transformation of (N, w) and thus a multiple of x is a balanced value of (N, v) by Proposition 3(i). By component efficiency, x itself is a balanced value of (N, v).

Now assume that $x \in B(N, v)$. Proposition 3(i) and the equality $w(N) = v(N)$ give $x \in B(N, w)$. Let $y \in F(N, w)$. By component efficiency of the balanced solution and of F we have $w(\{i\}) = x_i = y_i$ for every $i \in N \setminus Q$ and $w(Q) = x_Q = y_Q = 0$. Thus we get $x = y \in F(N, w)$. The game (N, v) is a balanced transformation of (N, w) and thus a multiple of x is in $F(N, v)$. By component efficiency, x itself is in $F(N, v)$ and we are done.

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