Coalitional Bargaining with Markovian Proposers

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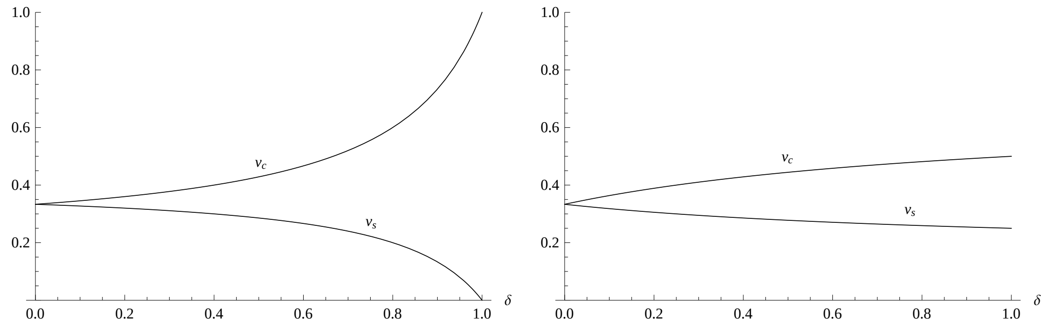
The paper generalizes coalitional bargaining of the Baron and Ferejohn-type (Eraslan and McLennan, 2006) to non-independent proposers. Widening the scope of possible dynamics significantly, this allows for likely continued offers, alternating offers (Calvó-Armengol, 2001b), and deterministic dynamics such as clockwise rotation of proposers (Herrero, 1985). Existence of stationary subgame perfect equilibria is shown as well as non-uniqueness of corresponding payoffs. Differences of two canonical protocols, the indepdenent and the alternating protocol, are discussed. Finally, we show that, for almost all protocols, veto players hold all bargaining power.

1 Introduction

In coalitional bargaining as first introduced in Baron and Ferejohn (1989), multiple players bargain non-cooperatively on how to split a unit surplus that can be generated only once by one of certain feasible coalitions. In every round of the game with infinite time horizon, one player is randomly assigned the role of a proposer who can offer a split of the value of 1 to one of the feasible coalitions. If all members of that coalition accept the proposer's offer, the surplus is split accordingly and the game ends. If anyone does yet reject the offer, the game proceeds to the next round and all possible future payoffs are discounted. Baron and Ferejohn (1989) only consider a simple majority rule defining the feasible coalitions, yet the model has successively been extended. Eraslan (2002) more generally allows for any kind of majority rule with symmetric voters, and Eraslan and McLennan (2006) implement arbitrary sets of feasible coalitions.



(a) Independent protocol (b) Alternating protocol

Figure 1: SSPE payoffs of example 1.1

This paper now sets out to extend coalitional bargaining in one additional way. Instead of further modifying the possible feasible coalitions, which has been exhaustively done, we relax one key assumption common to all above models, the assumption that proposers are chosen according to some random protocol and, in addition, independently in different rounds. While we still rely on, except for degenerated cases, random protocols, we waive the relatively strong assumption of independence. In a so to speak Markovian way, the probability distribution for the choice of a proposer is allowed to depend on the identity of the proposer as well as the coalition of the previous round.

Compared to coalitional bargaining with independent proposers, the scope of possible dynamics is widened significantly. The non-independence of proposers for instance allows for likely continued offers in the sense that a proposer can have increased chances of making a new offer after his current one is rejected. Not less plausible, respondents (the members of the proposed coalition, except for the proposer) can have greater than normal probability of making a counteroffer in the next round. Taking this to an extreme where only respondents have the chance of being proposer in the next round, our model incorporates the proposer dynamics of Calvó-Armengol (2001a,b). The two papers model bilateral bargaining among multiple players and as such are extensions of the alternating offer model of Rubinstein (1982). However, in contrast to our model, they only consider exogenous matching (Calvó-Armengol, 2001a) or deterministic, non-random selection of bargaining partners (Calvó-Armengol, 2001b). Finally, degenerated Markovian dynamics can implement deterministic protocols such as clock-wise rotation of proposers in the multilateral Rubinstein (1982) extension of Herrero (1985).

To see what qualitative impact the choice of protocol can have, compare the two canonical protocols as in the following example.

1.1 Example *Let there be three players c, s_1, and s_2 such that central player c can generate the unit surplus with either of the two minor players s_1 or s_2. Consider, on the one hand, the protocol where all players are chosen as the proposer in all rounds with equal probability (and thus, in fact, independently). On the other hand, as in Calvó-Armengol (2001a,b), let all players be proposer in the first round with equal probability* *and, in all subsequent rounds, the respondent of the previous round take the role of the proposer. In addition, let all players have a common unconditional discount factor δ, $0 \leq \delta < 1$. Figure 1.1 shows stationary subgame perfect equilibrium payoffs v_c of player c and v_s of players s_1 or s_2 as a function of δ. Both equilibria are symmetric in that player c proposes to both minor players s_1 and s_2 with equal probability. In the case of the independent protocol (figure 1(a)), player c holds all power (as which payoffs are usually interpreted when players grow infinitely patient, i.e. when $\delta \to 1$). Figure 1(b) shows that, however, he only holds half of the power if proposers are chosen according to the alternating protocol.*

The comparison shows a great sensitivity of the equilibrium payoffs with respect to modeling assumptions and raises the question of whether such qualitative differences persist in general and, if yes, in what way.

In addition, the example highlights the possibility of non-cooperative support for power indices in this extended framework. For the case of independent proposers, Nohn (2010) shows that veto players hold all bargaining power and share it proportional to their recognition probabilities, or hold no power at all. Non-cooperative support for power indices in a setting with independent proposers is hence restricted to power indices that assign all or no power to veto players. This, unfortunately, excludes support for popular indices such as the Shapley-Shubik index (Shapley and Shubik, 1954). For the given example, however, the Shapley-Shubik index amounts to $\frac{2}{3}$ for player c and $\frac{1}{6}$ for both minor players s_1 and s_2. It could hence be supported by a suitable protocol somewhere 'between' the independent and alternating protocol if only the limit of payoffs behaved 'sufficiently continuous' in the proposer dynamics.

The remainder of the paper is organized as follows. Section 2 defines our extension of coalitional bargaining games. A characterization of stationary subgame perfect equilibria, the solution concept we apply, is provided in section 3. Section 4 presents and proofs the existence of SSPE for all coalitional bargaining games. The non-uniqueness of corresponding payoffs, in contrast to the case of independent proposers (Eraslan and McLennan, 2006), is shown in section 5. Finally, section 6 provides a result for the power of veto players, saying they hold all power for almost all protocols.

2 The Coalitional Bargaining Game

Let N be the non-empty and finite *set of players*. For all $i \in N$, $W_i \subseteq 2^N$ with $W_i \neq \emptyset$ and $S \ni i$ for all $S \in W_i$ denotes player i's *set of feasible coalitions*. By $W = (W_i)_{i \in N}$ we refer as the *collection of feasible coalitions*. As Eraslan and McLennan (2006), we do not in principle require the sets of feasible coalitions to be related in any way. We call any pair (i, S) where $i \in N$ is a player and $S \in W_i$ one of player i's feasible coalitions a *match* and denote the set of matches by M. The *protocol* is a collection $p = (p^0, (p^{jT})_{(j,T) \in M})$ that defines the *recognition probabilities* for the assignment of proposers. In this, $p^0 = (p_i^0)_{i \in N}$ with $p^0 \geq 0$ and $\sum_i p_i^0 = 1$ is the *initial protocol* for the choice of a proposer in the first round. In any subsequent round, the recognition probabilities depend on the match of the previous round. We call $p^{jT} = (p_i^{jT})_{i \in N}$ with $p^{jT} \geq 0$ and $\sum_i p_i^{jT} = 1$ the *conditional* *protocol* for match $(j,T) \in M$.[1] In addition, players have a common discount factor δ according to which future payoffs are discounted in case of a delay of one round.[2][3]

A *coalitional bargaining game* is defined by the tuple (N, W, p, δ) containing the set of players N, the set of feasible coalitions W, the protocol p, and the discount factor δ. The game has an infinite time-horizon such that bargaining takes place in rounds $0, 1, \ldots$. Players are risk-neutral, so without loss of generality we identify their utility with the share they receive. Finally, players have complete and perfect information, and everything is common knowledge. The game is then played as follows. In any round, a player $i \in N$ is chosen to be the *proposer* for which the recognition probabilities are given by the initial protocol $p^0 = (p_i^0)_{i \in N}$ in the first round or by conditional protocols $p^{jT} = (p_i^{jT})_{i \in N}$ after rounds with match $(j,T) \in M$, respectively. Proposer i selects one of his feasible coalitions $S \in W_i$. He suggests a division of the surplus of 1 between him and the *respondents* $k \in S \backslash \{i\}$. The respondents decide in some arbitrary order - which one is irrelevant - on the proposal's acceptance or rejection.[4] If at least one respondent rejects the offer, the game proceeds to the next round and all future payoffs are discounted according to the discount factor δ. Unanimous acceptance, however, leads to the respective division and terminates the game.

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[1] As a natural approach to non-independent protocols, consider that players attempt to make offers with rates depending on their current role in bargaining. More specifically, players independently attempt to make offers after exponentially distributed waiting times, and the player trying first initiates the next round as the proposer. For each player $i \in N$, let $(o_i^0, o_i^p, o_i^r, o_i^n)$ denote the nonnegative *offering rates* for the first round, for previously being proposer, respondent, or not being involved in bargaining, respectively (in case a rate is 0, the respective player does not make offers in that particular situation). Assume the initial total offering rate $o^0 = \sum_{i \in N} o_i^0$ as well as all conditional total offering rates $o^{jT} = o_j^p + \sum_{k \in T \backslash \{j\}} o_k^r + \sum_{k \in N \backslash T} o_k^n$, $(j,T) \in M$, are positive. Then, in the first round, $p_i^0 = o_i^0 / o^0$, $i \in N$. For all rounds following a match $(j,T) \in M$, $p_i^{jT} = o_i^p / o^{jT}$ for previous proposer $i = j$, $p_i^{jT} = o_i^r / o^{jT}$ for previous respondents $i \in T \backslash \{j\}$, and $p_i^{jT} = o_i^n / o^{jT}$ for players $i \notin T$ not previously involved in bargaining.

[2] It is equally possible to assume individual and conditional discount factors. One then has a collection of discount factors $(\delta_i^{jT})_{i \in N, (j,T) \in M}$ such that the payoff of player $i \in N$ is discounted by δ_i^{jT} after any round with match $(j,T) \in M$. A collection of conditional discount factors thus allows to incorporate individual time preferences and delays that depend for instance on the size of the coalition. While conditional discount factors might thus have a significant impact on the selection behavior of proposers, we do not incorporate them in the model due to a lack of respective analytical statements. However, all statements of sections 3 and 4 can be modified accordingly and proven analogously without any further effort.

[3] With respect to footnotes 1 and 2, assume that players $i \in N$ have offering rates $(o_i^0, o_i^p, o_i^r, o_i^n)$ and *unit discount factors* δ_i, $0 \leq \delta_i < 1$, for the discount of payoffs in a unit time interval. For convention, let $ln(0) = -\infty$ and $1/\infty = 0$. Conditional discount factors then arise as the expected discount between two rounds and amount to $\delta_i^{jT} = o^{jT} / (o^{jT} - ln(\delta_i))$, $(j,T) \in M$, $i \in N$.

[4] It is crucial that respondents decide in some arbitrary order and not simultaneously. The latter case allows for two or more respondents jointly imposing arbitrary threats to the proposer as in Haller (1986).

3 Characterization of Stationary Subgame Perfect Equilibra

Since games of this kind typically have a multiplicity of equilibria and corresponding payoffs (see for instance the simple majority case in Baron and Ferejohn, 1989), we follow the mainstream literature and restrict attention to stationary strategies. Strategies of players are *stationary* whenever actions are both *time-homogeneous* and *independent* of previous rounds. Thus, strategies are fully described as follows. In any round, a player $i \in N$ chosen as the proposer selects a feasible coalition $S \in W_i$ according to a stationary *selection distribution* $\lambda_i = (\lambda_{iS})_{S \in W_i}$. Conditional on having selected coalition $S \in W_i$, he suggests a split $x^{iS} = (x_k^{iS})_{k \in S}$ of the surplus of 1 between the members of that coalition. The decisions about acceptance or rejection of respondents $k \in S \backslash \{i\}$ depend only on the offer itself and the current match $(i, S) \in M$.

Strategies being stationary, also the payoffs players expect before any round are stationary with respect to the match of last round. Given current match $(j, T) \in M$, denote by $v^{jT} = (v_i^{jT})_{i \in N}$ the conditional payoffs of players before next round. By $v^0 = (v_i^0)_{i \in N}$ we denote the payoffs players expect before the game. Stationarity allows to characterize the strategies by the corresponding conditional payoffs.

3.1 Proposition *SSPE strategies are characterized by the corresponding conditional payoffs* $(v^{jT})_{(j,T) \in M}$ *as follows.*

(i) *Any player* $i \in N$ *uses a selection distribution* λ_i *with positive probability only on coalitions* $S \in W_i$ *which maximize* $1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS}$.

(ii) *Having chosen coalition* $S \in W_i$*, proposer* $i \in N$ *offers any respondent* $k \in S \backslash \{i\}$ *that player's* continuation value*,* $x_k^{iS} = \delta v_k^{iS}$*, his own share amounting to* $x_i^{iS} = 1 - \sum_{k \in S \backslash i} \delta v_k^{iS}$.

(iii) *Given current match* $(i, S) \in M$*, respondents* $k \in S \backslash \{i\}$ *accept any share* x_k^{iS} *not exceeded by their continuation value* δv_k^{iS}.[5]

Proof

(i) Follows given later behavior as described by (ii) and (iii).

(ii) Consider a proposer $i \in N$ making an offer to coalition $S \in M_i$. Given respondents' behavior according to (iii), offering $x_k^{iS} = \delta v_k^{iS}$ to all respondents $k \in S \backslash \{i\}$ and keeping $x_i^{iS} = 1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS}$ for himself is the most profitable among all accepted offers. Moreover, due to $1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS} > \delta v_i^{iS}$, making this offer is strictly better than continuation of the game in the next round due to a rejected offer.

(iii) Consider proposer $i \in N$ suggesting split $(x_k^{iS})_{k \in S}$ to coalition $S \in W_i$. Respondents decide in some arbitrary order on the proposal's acceptance. In case that any respondent rejects, respondent $k \in S \backslash \{i\}$ alternatively expects δv_k^{iS} by continuation of the game in the next round. So acceptance (rejection) is a best response for respondent

[5] Note there are SSPE in which respondents decide differently. This is possible, however, only in situations where another respondent already has, or certainly will, reject an offer. Hence, all these SSPE are equivalent in outcome and payoffs.

$k \in S\backslash\{i\}$ in case that $x_k^{iS} > \delta v_k^{iS}$ ($x_k^{iS} < \delta v_k^{iS}$) and it then is a strictly best action if and only if all previous and subsequent respondents accept. Being offered his reservation value, i.e. $x_k^{iS} = \delta v_k^{iS}$, the respondent is indifferent and can well accept (and, in fact, acceptance in this case is necessary for the existence of a SSPE). $\square$

Note that, unlike in the case with independent proposers, proposer $i \in N$ typically does not choose a coalition $S \in W_i$ with maximum *excess* $1 - \sum_{k\in S} \delta v_k^{iS}$. Since his continuation value δv_i^{iS} possibly depends on S, maximization of the excess $1-\sum_{k\in S} \delta v_k^{iS}$ is not necessarily equivalent to the maximization of his share $1 - \sum_{k\in S\backslash\{i\}} \delta v_k^{iS}$. Thus, depending on the protocol, it is even possible that players propose to coalitions which are not minimal.[6]

By proposition 3.1, any stationary subgame perfect equilibbrium is fully described by its conditional payoffs $(v^{jT})_{(j,T)\in M}$ and selection distributions $(\lambda_i)_{i\in N}$. However, it remains to show which conditional payoffs and selection distributions are supported by a SSPE.

3.2 Proposition *A collection of conditional payoffs $(v^{jT})_{(j,T)\in M}$ is supported by a SSPE with selection distributions $(\lambda_i)_{i\in N}$, if and only if*

for all matches $(j,T) \in M$ and all $i \in N$ it holds

$$v_i^{jT} = p_i^{jT} \sum_{S\in W_i} \lambda_{iS}(1 - \sum_{k\in S\backslash\{i\}} \delta v_k^{iS}) + \sum_{k\in N\backslash\{i\}} p_k^{jT} \sum_{S\in W_k, S\ni i} \lambda_{kS} \delta v_i^{kS} \tag{3.1}$$

and for all $i \in N$ it holds that λ_i maximizes

$$\sum_{S\in W_i} \lambda_{iS}(1 - \sum_{k\in S\backslash\{i\}} \delta v_k^{iS}) \tag{3.2}$$

given constraints $\lambda_i \geq 0$ and $\sum_{S\in W_i} \lambda_{iS} = 1$.

Proof

(Only if.) Let conditional expectations $(v^{jT})_{(j,T)\in M}$ and selection distributions $(\lambda_i)_{i\in N}$ be supported by an SSPE. Then strategies according to proposition 3.1 yield conditional payoffs as in (3.1) and selection distributions satisfying (3.2).

(If.) Assume $(v^{jT})_{(j,T)\in M}$ and $(\lambda_i)_{i\in N}$ satisfy (3.1) and (3.2) and consider strategies as follows. Let selection distributions be given by $(\lambda_i)_{i\in N}$. Given player $i \in N$ proposes to coalition $S \in W_i$, let him offer δv_k^{iS} to any respondent $k \in S\backslash\{i\}$ and keep the remaining $1 - \sum_{k\in S\backslash\{i\}} \delta v_k^{iS}$ for himself. Let respondents $k \in S\backslash\{i\}$ accept any share not exceeded by their reservation value δv_k^{iS}. Due to (3.1), these strategies generate conditional payoffs again given by $(v^{jT})_{(j,T)\in M}$. Thus, from (3.2) and proposition 3.1 it follows that these strategies constitute a SSPE. $\square$

[6] Consider $N \in W$ and a protocol p with $p_i^{iN} = 1$, $i \in N$. In this case, any player would always propose to the grand coalition.

Note that, for given selection distributions, conditional payoffs correspond in a one-to-one fashion with the offers made. Conditional payoffs v_i^{jT} where $(j,T) \in M$ and $i \notin T\backslash\{j\}$ are hence redundant in the equilibrium characterization and one can restrict attention to the conditional payoffs of previous respondents, v_i^{jT} for which $(j,T) \in M$ and $i \in T\backslash\{j\}$. Thus, a characterization based on conditional payoffs is equivalent to a characterization based on offers (as in Calvo-Armengol, 2001a,b).

4 Existence of Stationary Subgame Perfect Equilibra

The propositions of the previous section allow for the following statement.

4.1 Theorem *In any coalitional bargaining game (N, W, p, δ), there exists a stationary subgame perfect equilibrium.*

Proof We prove the theorem in three stages. We first show that (i) for any given collection of selection distributions $(\lambda_i)_{i\in N}$, there is one unique solution $(v^{jT})_{(j,T)\in M}$ to equation system (3.1). Then, (ii), any such solution constitutes a collection of nonnegative and normalized vectors and is hence feasible as a collection of conditional payoffs. Finally, (iii), Kakutani's fixed-point theorem is applied to show the mutual existence of $(v^{jT})_{(j,T)\in M}$ and $(\lambda_i)_{i\in N}$ such that (3.1) and (3.2).

(i) For a given collection of selection distributions $(\lambda_i)_{i\in N}$, assume there are two distinct solutions $(v^{jT})_{(j,T)\in M}$ and $(w^{jT})_{(j,T)\in M}$ to equation system (3.1). For any match $(k,S) \in M$, denote $N_+^{kS} = \{i \in N | v_i^{kS} - w_i^{kS} > 0\}$ and fix one match $(j,T) \in M$ such that $\sum_{i\in N_+^{jT}}(v_i^{jT} - w_i^{jT})$ is maximal among all $\sum_{i\in N_+^{kS}}(v_i^{kS} - w_i^{kS})$, $(k,S) \in M$. In particular, it holds

$$\sum_{i\in N_+^{jT}} (v_i^{jT} - w_i^{jT}) > \sum_{i\in S\cap N_+^{jT}} \delta(v_i^{kS} - w_i^{kS})$$

for all $(k,S) \in M$. Also note that from (3.1) it follows $\sum_{i\in N} v_i^{kS} = 1$ for all $(k,S) \in M$. Thus, $\sum_{i\in N}(v_i^{kS} - w_i^{kS}) = 0$ for all $(k,S) \in M$ and hence $\sum_{i\in N_+^{jT}}(v_i^{jT} - w_i^{jT}) \geq -\sum_{i\in N\backslash N_+^{kS}}(v_i^{kS} - w_i^{kS})$ for all $(k,S) \in M$. In particular,

$$\sum_{k\in N_+^{jT}} (v_k^{jT} - w_k^{jT}) > -\sum_{k\in S\backslash N_+^{jT}} \delta(v_k^{iS} - w_k^{iS})$$

for all $(i,S) \in M$. Thus, summing up equation system (3.1) over $i \in N_+^{jT}$ for both v and

w and considering the difference, we find

$$\begin{aligned}
&\sum_{i \in N_+^{jT}} (v_i^{jT} - w_i^{jT}) \\
=& - \sum_{i \in N_+^{jT}} \sum_{S \in W_i} p_i^{jT} \lambda_{iS} \sum_{k \in S \setminus N_+^{jT}} \delta(v_k^{iS} - w_k^{iS}) \\
&+ \sum_{k \in N \setminus N_+^{jT}} \sum_{S \in W_k} p_k^{jT} \lambda_{kS} \sum_{i \in S \cap N_+^{jT}} \delta(v_i^{kS} - w_i^{kS}) \\
<& \sum_{i \in N_+^{jT}} \sum_{S \in W_i} p_i^{jT} \lambda_{iS} \sum_{k \in N_+^{jT}} (v_k^{jT} - w_k^{jT}) \\
&+ \sum_{k \in N \setminus N_+^{jT}} \sum_{S \in W_k} p_k^{jT} \lambda_{kS} \sum_{i \in N_+^{jT}} (v_i^{jT} - w_i^{jT}) \\
=& \sum_{i \in N_+^{jT}} (v_i^{jT} - w_i^{jT}),
\end{aligned}$$

a contradiction. Hence, linear equation system (3.1) always has a unique solution.

(ii) Consider any given collection of selection distributions $\lambda = (\lambda_i)_{i \in N}$. For any match $(j, T) \in M$ and player $i \in N$, define the mapping f_i^{jT} by

$$\begin{aligned}
&f_i^{jT}((x_{i'}^{j'T'})_{(j',T') \in M, i' \in N}) \\
&= p_i^{jT} \sum_{S \in W_i} \lambda_{iS} (1 - \sum_{k \in S \setminus \{i\}} \delta x_k^{iS}) + \sum_{k \in N \setminus \{i\}} p_k^{jT} \sum_{S \in W_k, S \ni i} \lambda_{kS} \delta x_i^{kS}.
\end{aligned}$$

In case that $(x_{i'}^{j'T'})_{i' \in N}$ is a nonnegative and normalized vector for any match $(j', T') \in M$, then also $(f_i^{jT}((x_{i'}^{j'T'})_{(j',T') \in M, i' \in N})_{i \in N}$ is nonnegative and normalized for any match $(j, T) \in M$. From f's continuity and Brouwer's fixed-point theorem, it follows the existence of a collection $(v_i^{jT})_{(j,T) \in M, i \in N}$ of nonnegative and normalized vectors which constitutes a fixed-point of $(f_i^{jT})_{(j,T) \in M, i \in N}$ or, equivalently, which satisfies linear equation system (3.1).

(iii) For any given collection of selection distributions $\lambda = (\lambda_i)_{i \in N}$, denote by $\bar{v}^{jT}(\lambda) = (\bar{v}_i^{jT}(\lambda))_{i \in N}$ for every match $(j, T \in M)$ the corresponding conditional payoff vector resulting from linear equation system (3.1). For any collection of conditional payoff vectors $(v^{jT})_{(j,T) \in M}$, denote by $\bar{\lambda}((v^{jT})_{(j,T) \in M})$ the set of collections of selection distributions solving optimization problem (3.2). By the theorem of the maximum, $\bar{\lambda}((v^{jT})_{(j,T) \in M})$ is upper semi-continuous in $(v^{jT})_{(j,T) \in M}$. Since $(\bar{v}^{jT}(\lambda))_{(j,T) \in M}$ is continuous, also $\bar{\lambda}((\bar{v}^{jT}(\lambda))_{(j,T) \in M})$ is upper semi-continuous. Kakutani's fixed-point theorem ensures the existence of a collection of selection distributions λ^* with $\lambda^* \in \bar{\lambda}((\bar{v}^{jT}(\lambda^*))_{(j,T) \in M})$. By construction, the pair $(\bar{v}^{jT}(\lambda^*))_{(j,T) \in M}$ and λ^* satisfy linear equation system (3.1) and optimization problem (3.2) and hence are, by proposition 3.2, supported by a SSPE. $\square$

The *ex-ante* payoffs v^0 in a SSPE with conditional payoffs $(v^{jT})_{(j,T)\in M}$ and selection distributions $\lambda = (\lambda_i)_{i\in N}$ then amount, for $i \in N$, to

$$v_i^0 = p_i^0 \sum_{S\in W_i} \lambda_{iS}(1 - \sum_{k\in S\setminus\{i\}} \delta v_k^{iS}) + \sum_{k\in N\setminus\{i\}} p_k^0 \sum_{S\in W_k, S\ni i} \lambda_{kS}\delta v_i^{kS}. \tag{4.1}$$

5 Non-Uniqueness of Stationary Equilibrium Payoffs

Eraslan and McLennan (2006) show that, while there may be a multiplicity of stationary subgame perfect equilibria, the corresponding (ex-ante) payoffs are unique in the case of independent proposers. In the general case of possibly dependent proposers, we find this uniqueness not to hold anymore. To see this, consider the bargaining situation from the introductory example with an arbitrary number of players and alternating protocol.

In general, the *alternating protocol* can only be considered for games with no feasible singleton coalition, i.e. $\{i\} \notin W_i$ for all $i \in N$. Initially, it assigns equal probability to be the proposer to all players and, in all subsequent rounds, the respondents of the previous round have equal probability to be the proposer. Formally, $p_i = 1/|N|$ for all $i \in N$ and for all $(j, T) \in M$, $p_i^{jT} = 1/(|T| - 1)$ for $i \in T\setminus\{j\}$ and $p_i^{jT} = 0$ for $i \notin T$.

5.1 Example *Let there be $1 + n_s \geq 2$ players, $N = \{c, s_1, \ldots, s_{n_s}\}$, such that central player c can create a surplus of 1 together with any single minor player $s_1, \ldots, s_{n_s}$. The feasible coalition are hence given by $S_i = \{c, s_i\}$, $i = 1, \ldots, n_s$. Let proposers be assigned according to the alternating protocol and let players have a discount factor δ, $0 \leq \delta < 1$. Restricting attention to the conditional payoffs of previous respondents, the particular instances of equation system (3.1) arise to*

$$v_c^{s_iS_i} = \sum_{j=1,\ldots,n_s} \lambda_{cS_j}(1 - \delta v_{s_j}^{cS_j}), \quad i = 1, \ldots, n_s, \tag{5.1}$$

$$v_{s_i}^{cS_i} = 1 - \delta v_c^{s_iS_i}, \quad i = 1, \ldots, n_s. \tag{5.2}$$

Conditional payoffs thus happen to be independent of player c's selection distribution $\lambda_c = (\lambda_{cS_1}, \ldots, \lambda_{cS_{n_s}})$ and, more precisely, the conditional payoff, or share, of any proposer is that of the proposer in a 2-player Rubinstein alternating offer game (Rubinstein, 1982),

$$v_c^{s_iS_i} = \frac{1}{1+\delta}, \quad i = 1, \ldots, n_s, \tag{5.3}$$

$$v_{s_i}^{cS_i} = \frac{1}{1+\delta}, \quad i = 1, \ldots, n_s. \tag{5.4}$$

In particular, there is an SSPE for every possible selection distribution λ_c of player c. Unlike in the case with independent proposers, ex-ante SSPE payoffs v^0 coincide only if players are myopic,

$$v_c^0 = \frac{1 + n_s\delta}{(1 + n_s)(1 + \delta)}, \tag{5.5}$$

$$v_{s_i}^0 = \frac{1 + \lambda_{cS_i}\delta}{(1 + n_s)(1 + \delta)}, \quad i = 1, \ldots, n_s. \tag{5.6}$$

Interestingly, the multiplicity of stationary equilibrium payoffs found for non-zero discount factors seems to be a mere singularity. If the alternating protocol is slightly perturbed, the selection probabilities of player c do not drop out in equations (5.1) and (5.2) and only a single selection distribution of player c constitutes an equilibrium.

6 The Power of Veto Players

Given non-empty and finite set of players N, set of feasible coalitions W and protocol p, denote by $\bar{\xi}$ the non-empty and compact set of all limit points of ex-ante SSPE payoffs in coalitional bargaining games (N, W, p, δ) as δ approaches 1. Very commonly, a vector $\xi = (\xi_i)_{i \in N}$ that lies in $\bar{\xi}$ is interpreted as *(bargaining) power* for set of players N, set of feasible coalitions W and protocol p. In the case of unique SSPE payoffs such as with independent proposers (Eraslan and McLennan, 2006), it follows that $\bar{\xi}$ contains a single element only. We say that $\xi \in \bar{\xi}$ is *obtained* by a collection of selection distributions $\lambda = (\lambda_i)_{i \in N}$ if there is a sequence of SSPE in games (N, W, p, δ) that have collections of selection distributions converging to λ and payoffs converging to ξ as δ approaches 1.

A *veto player* $i \in N$ is a member of all feasible coalitions, $i \in S$ for all $S \in W_j$ for all $j \in N$, and is thus able to block all proposals. Denote V as the set of veto players. For the case of independent proposers and some veto player having positive recognition probability, Nohn (2010) shows that veto players hold all power. The following proposition generalizes this result to a far weaker condition.

6.1 Proposition *Let N be a non-empty and finite set of players, W a collection of feasible coalitions, and p a protocol. Assume the set of veto players V is non-empty. Further let there be a veto player $i^* \in V$ such that, firstly, for all $j \notin V$ there is $S \in W_{i^*}$ with $S \not\ni j$ and, secondly, $p_{i^*}^{jT} > 0$ for all $(j, T) \in M$.*[7] *It then holds for all $\xi \in \bar{\xi}$ that $\sum_{i \in V} \xi_i = 1$ and, in particular, $\xi_i = 0$ for all $i \notin V$.*

Proof Consider $\xi \in \bar{\xi}$ obtained by $\lambda = (\lambda_i)_{i \in N}$ and denote by $\xi^{jT} = (\xi_i^{jT})_{i \in N}$ the corresponding limit of conditional payoffs after match $(j, T) \in M$. We prove the proposition in seven steps.

(i) With ξ^{jT}, $(j, T) \in M$, and $\lambda = (\lambda_i)_{i \in N}$ arising as limits of conditional payoffs and selection distributions of SSPE, proposition 3.2 yields for all matches $(j, T) \in M$ and all $i \in N$ that

$$\xi_i^{jT} = p_i^{jT} \sum_{S \in W_i} \lambda_{iS} (1 - \sum_{k \in S} \xi_k^{iS}) + \sum_{k \in N} p_k^{jT} \sum_{S \in W_k, S \ni i} \lambda_{kS} \xi_i^{kS}. \tag{6.1}$$

Also, for all $i \in N$ it holds that λ_i maximizes $\sum_{S \in W_i} \lambda_{iS} (1 - \sum_{k \in S \setminus \{i\}} \xi_k^{iS})$ given constraints $\lambda_i \geq 0$ and $\sum_{S \in W_i} \lambda_{iS} = 1$.

(ii) A match (i, S) is said to be (directly) *accessible* from match $(j, T) \in M$ if $p_i^{jT} \lambda_{iS} > 0$. Due to $p_{i^*}^{jT} > 0$ for all $(j, T) \in M$, all matches (i^*, S) where $S \in W_{i^*}$ and $\lambda_{i*S} > 0$

[7] Note that Nohn (2010) only considers feasible coalitions which are related in the sense that for all $i \in N$ it holds for all $S \in W_i$ that also $S \in W_j$ for all $j \in S$. With this, the former condition of no non-veto player being able to block all proposals of a veto player is always satisfied.

are directly accessible from any other match. Let M^* be the set of all matches that are directly or indirectly accessible from matches (i^*, S), $S \in W_{i^*}$. All matches in M^* are at least indirectly accessible from each other.

(iii) Consider $i \in V$ and choose $(j,T) \in M^*$ such that ξ_i^{jT} is minimal among all ξ_i^{kS}, $(k,S) \in M^*$. Due to $\sum_{k \in N} p_k^{jT} \sum_{S \in W_k} \lambda_{kS} = 1$, equation (6.1) requires $\xi_i^{kS} = \xi_i^{jT}$ for all $(k,S) \in M^*$ that are accessible from (j,T). Iteration yields $\xi_i^{kS} = \xi_i^{jT}$ for all $(k,S) \in M^*$.

(iv) Consider $i \in V$ and choose $(j,T) \in M$ such that ξ_i^{jT} is minimal among all ξ_i^{kS}, $(k,S) \in M$. Assume $\xi_i^{jT} < \xi_i^{kS}$ for $(k,S) \in M^*$. Note that $p_{i^*}^{jT} \sum_{S \in W_{i^*}} \lambda_{i^*S} > 0$ and $(i^*, S) \in M$ for all $S \in W_{i^*}$ with $\lambda_{i*S} > 0$. Thus we have $\sum_{k \in N} p_k^{jT} \sum_{S \in W_k} \lambda_{kS} \xi_i^{kS} > \xi_i^{jT}$, a contradiction to equation (6.1). Thus, $\xi_i^{jT} \geq \xi_i^{kS}$ for $(k,S) \in M^*$.

(v) For $i \in V$ for which there is $(j,T) \in M^*$ with $p_i^{jT} > 0$, (iii) and equation (6.1) require $\sum_{S \in W_i} \lambda_{iS}(1 - \sum_{k \in S} \xi_k^{iS}) = 0$. Given optimality of λ_i, this is equivalent to $\sum_{k \in S} \xi_k^{iS} = 1$ for all $S \in W_i$.

(vi) Let $i \notin V$. By (v), $\xi_i^{i^*S} = 0$ for all $S \in W_{i^*}$ for which $i \notin S$. Note there is at least one such coalition $S \in W_{i^*}$. Equation (6.1) yields $\sum_{k \in N} p_k^{i^*S} \sum_{S \in W_k, S \ni i} \lambda_{kS} \xi_i^{kS} \leq \xi_i^{i^*S} = 0$. This requires $\xi_i^{jT} = 0$ for all $(j,T) \in M^*$ accessible from (i^*, S) and for which $T \ni i$. In particular, $\xi_i^{i^*T} = 0$ for all $T \in W_{i^*}$ where $(i^*, T) \in M^*$ and $T \ni i$.

(vii) From (vi), $\xi_i^{i^*S} = 0$ for all $i \notin V$, $S \in W_{i^*}$ where $(i^*, S) \in M^*$. Thus, $\sum_{i \in V} \xi_i^{i^*S} = 1$. Due to (iii), $\sum_{i \in V} \xi_i^{jT} = 1$ for all $(j,T) \in M^*$. By (iv), $\sum_{i \in V} \xi_i^{jT} = 1$ for all $(j, T \in M)$. Thus, for the ex-ante power of veto players, $\sum_{i \in V} \xi_i = 1$, and $\xi_i = 0$ for all $i \notin V$. □

Proposition 6.1 states that veto players hold all power if there is a veto player whose offers cannot all be blocked by any single non-veto player and who always has positive recognition probability.

Unlike in the mentioned case of independent proposers, however, we are not able, at least not at this point, to specify how exactly power is distributed among veto players. Inferring from Nohn (2010) and Britz et al. (2010), one might yet have a rather strong conjecture about the specific distribution of power. Nohn (2010) shows that, with independent proposers and some veto player having positive recognition probability, bargaining between as players grow infinitely patient bargaining essentially turns into unanimity bargaining between the veto players only who then share all power proportional to their recognition probabilities. Britz et al. (2010) consider unanimity bargaining with non-transferable utility where an irreducible and aperiodic Markov chain defines the proposers. The stationary distribution of this process then materializes as the weights of the asymmetric Nash solution to which payoffs converge as the risk of breakdown vanishes. For our case of transferable utility, this means that payoffs in unanimity bargaining are given by the stationary distribution of proposers. Now, consider the following. Given a collection of selection distributions $\lambda = (\lambda_i)_{i \in N}$, denote transition probabilities of proposers by $p_j^i = \sum_{S \in W_i} \lambda_{iS} p_j^{iS}$, $i, j \in N$. Under the conditions of proposition 6.1, the Markov chain defined by $(p_j^i)_{i,j \in N}$ is aperiodic and irreducible on exactly one subset of players $N^* \subseteq N$. It hence has a *stationary protocol* $\pi = (\pi_i)_{i \in N}$ uniquely defined by

$\pi_i = \sum_j \pi_j p_i^j$ and $\pi \geq 0$. In the long run, and given that all offers are rejected, all players $i \in N$ are proposer in a relative share of π_i of all rounds.[8]

6.2 Conjecture *Let N be a non-empty and finite set of players, W a collection of feasible coalitions, and p a protocol. Assume the conditions of proposition 6.1 to hold and let $\xi \in \bar{\xi}$ be a power distribution obtained by $\lambda = (\lambda_i)_{i \in N}$. Let $(p_j^i)_{i,j \in N}$ denote the corresponding transition probabilities of proposers with stationary protocol $\pi = (\pi_i)_{i \in N}$. It then holds $\xi_i = \pi_i / \sum_{j \in V} \pi_j$ for all $i \in V$, and, in particular, $\xi_i = 0$ for all $i \notin V$.*

As seen earlier, one case where veto players do not hold all power is the star with alternating protocol (example 5.1). This result can be stated more generally.

6.3 Example *Let there be $n_c \geq 1$ central players and $n_s \geq 1$ small players, $N = \{c_1, \dots, c_{n_c}, s_1, \dots, s_{n_s}\}$, such that all central players $c_1, \dots, c_{n_c}$ can create a surplus of 1 together with any single minor player $s_1, \dots, s_{n_s}$. Feasible coalitions are thus given by $S_i = \{c_1, \dots, c_{n_c}, s_i\}$, $i = 1, \dots, n_s$. Let proposers be assigned according to the alternating protocol and assume symmetric selection behavior of the central players, $\lambda_{c_i S_j} = 1/n_s$ for all $i = 1, \dots, n_c$ and $j = 1, \dots, n_s$. Then all minor players combined hold as much power as any single central player, $\xi_{c_i} = 1/(n_c+1)$ for all $i = 1, \dots, n_c$ and $\xi_{s_i} = 1/(n_s(n_c+1))$ for all $i = 1, \dots, n_s$.*

However, there are bargaining situations where also the alternating protocol assigns all power to a veto player.

6.4 Example *Let there be $1 + n_s \geq 4$ players, $N = \{c, s_1, \dots, s_{n_s}\}$, such that central player c can create a surplus of 1 together with any $n_s - 1$ minor players $s_1, \dots, s_{n_s}$. Feasible coalitions hence are $S_i = N \backslash \{s_i\}$, $i = 1, \dots, n_s$. Let proposers be assigned according to the alternating protocol and assume symmetric selection behavior of the central player, $\lambda_{cS_i} = 1/n_s$ for all $i = 1, \dots, n_s$, and as well of all minor players, $\lambda_{s_i S_j} = 1/(n_s - 1)$ for $i, j = 1, \dots, n_s$ where $i \neq j$. Then central player c holds all power, $\xi_c = 1$ and $\xi_{s_i} = 0$ for all $i = 1, \dots, n_s$.*

While in this case he needs the support of all but one minor players to create the surplus of 1, cooperation with any single minor player is sufficient to do so in the star. That veto player c does nevertheless hold all power in this example but not in the star with alternating protocol is a peculiarity of the alternating protocol for which we do not yet have, unfortunately, an intuitive explanation.

To summarize, protocols that do not assign all power to veto players are a mere singularity – arbitrarily small perturbations of any protocol ensure that veto players hold all power. The possibility of non-cooperative support is hence, in addition to Nohn (2010), further limited and restricts to singular cases such as the alternating protocol. Example 6.4 does yet show that even such protocols can fail in assigning non-zero power to non-veto players.

[8] For general information on aperiodic and irreducible Markov chains and stationary distributions, see Karlin and Taylor (1975).

References

Baron, D., Ferejohn, J., 1989. Bargaining in legislatures. American Political Science Review 83, 1181–1206.

Britz, V., Herings, P., Predtetchinski, A., 2010. Non-cooperative support for the asymmetric Nash bargaining solution. Journal of Economic Theory 145, 1951–1967.

Calvó-Armengol, A., 2001a. Bargaining power in communication networks. Mathematical Social Sciences 41, 69–87.

Calvó-Armengol, A., 2001b. On bargaining partner selection when communication is restricted. International Journal of Game Theory 30, 503–515.

Eraslan, H., 2002. Uniqueness of stationary equilibrium payoffs in the Baron-Ferejohn model. Journal of Economic Theory 103, 11–30.

Eraslan, H., McLennan, A., 2006. Uniqueness of Stationary Equilibrium Payoffs in Coalitional Bargaining. Working paper.

Haller, H., 1986. Non-cooperative bargaining of $N \geq 3$ players. Economics Letters 22, 11–13.

Herrero, M., 1985. N-player bargaining and involuntary unemployment. Ph.D. thesis, London School of Economics, London.

Karlin, S., Taylor, H., 1975. A First Course in Stochastic Processes. Academic Press, San Diego.

Nohn, A., 2010. Veto Players and Non-Cooperative Foundations of Power in Coalitional Bargaining. Working paper.

Rubinstein, A., 1982. Perfect Equilibrium in a Bargaining Model. Econometrica 50, 97–111.

Shapley, L., Shubik, M., 1954. A Method for Evaluating the Distribution of Power in a Committee System. American Political Science Review 48, 787–792.