MANIPULATION IN VOTING: STRATEGIC VOTING AND STRATEGIC NOMINATION

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(RESEARCH IN PROGRESS, PRELIMINARY AND INCOMPLETE)

Abstract: In this paper the concepts of manipulation as strategic voting (misrepresentation of true preferences) and strategic nomination (by adding, or removing alternatives) are investigated. The connection between Arrow's and Gibbard-Satterthwaite theorems is discussed from the viewpoint of dilemma between dictatorship and manipulability.

Keywords: Arrow's theorem, dictatorship, Gibbard-Satterthwaite theorem, manipulation, strategic voting, strategic nomination
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1. Introduction

Considerable social choice literature exists regarding manipulability of voting procedures (Taylor and Pacelli, 2008). Manipulability is usually understood as misrepresenting voters' preferences to get more beneficial outcome of voting. In this paper we distinguish between two kinds of manipulation:

Strategic voting (Gibbard 1973, Satterthwaite 1975, Gärdenfors 1979): On the basis of an information (or a hypothesis) about rankings of other voters and corresponding social rankings (defined by used voting rule) the voter submits such ranking, that maximizes her "utility" from resulting social ranking.

Strategic nomination (e.g. Tideman 1987): If the set of alternatives is endogenous (i.e. not fixed by nature), then outcomes can be manipulated by adding alternatives to or removing alternatives from the set of alternatives being voted upon.

Two famous social choice theorems are related to the problems of dictatorship and manipulability. While the Arrow's "impossibility" theorem is usually associated with non-existence of non dictatorial social preference function, the Gibbard-Satterthwaite theorem shows

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that any non-dictatorial non-degenerate social choice function is manipulable. In fact, many authors observe that the both theorems are closely related (Reny, 2000). In this paper we try to reformulate Arrow's and Gibbard-Satterthwaite theorems from the viewpoint of dilemma between dictatorship and manipulability.

2. Models of voting and manipulation

By voting we mean the following pattern of collective choice: There is a set of alternatives and a group of individuals. Individual preferences over the alternatives are exogenously specified and are supposed to be orderings. The group is required to choose an alternative on the basis of stating and aggregating of individual preferences, or to produce a ranking of alternatives from the most preferred to the least preferred.

2.1 Voting problem

Let U denotes a finite set, then by $\Pi(U)$ we denote the set of strict linear orders, or (strict) rankings, on U, by $\Pi^*(U)$ we denote the set of weak linear orders, or (weak) rankings, on U.

Let N denotes the set of n individuals (voters), U a universe of alternatives (finite set of cardinality m), and $Z \subseteq U$ is a subset of U of cardinality $t \leq m$. By $\Pi^n(Z)$ we denote n-fold Cartesian product of $\Pi(Z)$, and by $\Pi^{*n}(Z)$ n-fold Cartesian product of $\Pi^*(Z)$. An element

$$\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_n) \in \Pi^n(Z)$$

is called a preference profile on Z. A preference profile on Z is a set of individual preference relations π_i on Z with one and only one preference relation for each individual $i \in N$.

By voting problem we mean the following: given N, U, $\boldsymbol{\pi} \in \Pi^n(Z)$ and some set A of social choice rationality axioms, select $Z \in 2^U$ and for selected Z find: a) either social ordering $\pi_0 \in \Pi^*(Z)$ satisfying A, b) or $z_0 \in Z$ satisfying A.

If Z is fixed, a function $f: \Pi^N(Z) \to Z$ will be called a social choice function, while a function $F: \Pi^N(Z) \to \Pi^*(Z)$ will be called a social preference function.

2.2 Measuring distances between rankings

Representation of strict orderings: Let us consider set of alternatives Z = {x_1, x_2, …, x_m} and strict orderings $\pi_i \in \Pi(Z)$. Let $\Lambda(Z)$ be the set of all permutations of alternatives (1, 2, …, m), and $\lambda^i = (\lambda_{i1}, \lambda_{i2}, ..., \lambda_{im}) \in \Lambda$, then individual ranking $\pi_i = (x_{\lambda_{i1}}, x_{\lambda_{i2}}, ..., x_{\lambda_{im}})$, where $x_{\lambda_k}(\pi_i) = k$. Define Borda score vector b(π_i) = (m – x1(π_i), m – x2(π_i), …, xm(π_i))

Example 2.1: Z = {x1, x,2, x3, x4, x5}, λ^i = (2, 1, 3, 4, 5), π_i = [x2, x1, x3, x4, x5], x1(π_i) = 2, x2(π_i) = 1, x3(π_i) = 3, x4(π_i) = 4, x5(π_i) = 5, b(π_i) = (5 – x1(π_i), 5 – x2(π_i), 5 – x3(π_i), 5 – x4(π_i), 5 – x5(π_i) = (3, 4, 2, 1, 0).

Representation of weak orderings: Let us consider set of alternatives Z = {x_1, x_2, …, x_m} and weak orderings $\pi_0 \in \Pi^*(Z)$. A weak ordering is a partition Z = {Z1, Z2, …, Zv), v ≤ m, where Z1 is the set of first place alternatives, Z2 the set of second place alternatives etc. Set Z0 = {∅} and card (Z0) = 0. For xj ∈ Zk (k = 1, 2, ..., v) define average order in π_0

$$x_j(\pi_0) = \frac{\sum_{t=1}^{card(Z_k)} \left(t + \sum_{s=0}^{k-1} card(Z_s) \right)}{card(Z_s)}$$

Example 2.2: Z = {x1, x2, x3, x4, x5}, Z1 = {x1, x3}, Z2 = {x2}, Z3 = {x4, x5}, π_0 = [(x1, x3), x2, (x4, x5)], card (Z1) = 2, Z2 = 1, card (Z3) = 2,
x1(π_0) = x3(π_0) = (1+0)+(2+0))/2 = 1,5,
x2(π_0) = (1+2)/1 = 3,
x4(π_0) = x5(π_0) = ((1+3)+(2+3))/2 = 9/2 = 4,5,
hence average social ordering vector **x**(π_0) = (1,5, 3, 1,5, 4,5, 4,5)

Given a social preference function F, let F(**π,** Z) = $\pi_0 \in \Pi^*(Z)$. If π_i is i-th strict individual ranking and π_0 a weak social ranking, then we define distance between social ranking π_0 and individual ranking π_i as

$$d(\pi_i, \pi_0) = \sum_{j=1}^{m} abs(x_j(\pi_i) - x_j(\pi_0))$$

Example 2.3: Strict ranking π_i = [x2, x1, x3, x4, x5], weak ranking π_0 = [(x1, x3), x2, (x4, x5)]. Then **x**(π_1) = (2, 1, 3, 4, 5), **x**(π_0) = (1,5, 3, 1,5, 4,5, 4,5), and distance

$d(\pi_I, \pi_0)$ = abs (2-1,5)+abs(1-3)+abs(3-1,5)+abs(4-4,5)+abs (5-4,5)) = 0,5 + 2 + 1,5 + 0,5 + 0,5 = 5.

Example 2.4: Considering two different rankings π_0 = [(x1, x3), x2, x4] and π'_0 = [x2, x3, x1, x4, x5] and strict ranking π_i = [x2, x1, x3, x4, x5] we obtain
$d(\pi_i, \pi_0)$ = abs (2-1,5)+abs(1-3)+abs(3-1,5)+abs(4-4,5)+abs(5-4,5) = 0,5 + 2 + 1,5 + 0,5 + 0,5 = 5, $d(\pi_i, \pi'_0)$ = abs (2-3)+abs(1-1)+abs(3-2)+abs(4-4)+abs(5-5) = 1 + 0 + 1 + 0 + 0 = 2, hence weak ranking π'_0 is "closer" to the ranking π_i then the ranking π_0. If π_i is individual ranking of a i-th individual and π_0, π'_0 are two different social rankings, we can decide which of social rankings is "closer" to an individual ranking.

2.3 Social choice function

Let $z \in Z$ and $\boldsymbol{\pi} \in \Pi^n(Z)$, then by $z(\pi_i)$ we denote order number of alternative z in i'th individual ordering π_i (1 for top alternative, 2 for second alternative etc.), and by $\beta(z, \pi_i) = t - z(\pi_i)$ so called Borda score of z in the i'th voter's ranking.

We say that a social choice function $f(Z, \boldsymbol{\pi})$ has a property of:

Pareto efficiency if whenever alternative x is at the top of every individual i's ranking, π_i, then $f(Z, \boldsymbol{\pi}) = x$.

Monotonicity if whenever $f(Z, \boldsymbol{\pi}) = x$ and for every individual i and every alternative y the ranking π'_i ranks x above y if π_i does, then $f(Z, \boldsymbol{\pi}') = x$.

Dictatorship if there is an individual i such that $f(Z, \boldsymbol{\pi}) = x$ if and only if x is at the top of i's ranking πi.

Strategic voting manipulability if there exists a preference profile $\boldsymbol{\pi}$, a subset of individuals $M \subset N$ and a preference profile $\boldsymbol{\pi}'$ such that $\pi'_i = \pi_i$ for $i \in N\backslash M$, $f(Z, \boldsymbol{\pi}) = x$, $f(Z, \boldsymbol{\pi}') = y$, and for all $i \in M$ it holds that $y(\pi_i) < x(\pi_i)$.

Strategic nomination manipulability in Z if there exist Z' such that $Z \subset Z'$, a subset of individuals $M \subset N$ and preference profiles $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ where $\boldsymbol{\pi}'$ is an extension of $\boldsymbol{\pi}$ with the same individual preferences for Z, such that $f(Z, \boldsymbol{\pi}) = x \in Z$, $f(Z', \boldsymbol{\pi}') = y$, and for all $i \in N\backslash M$ it holds that $y(\pi_i) < x(\pi_i)$.

Non-degeneracy if for every $x \in Z$ there exist a preference profile $\boldsymbol{\pi} \in \Pi^n(Z)$ such that $f(Z, \boldsymbol{\pi}) = x$.

2.4 Social preference function

We say that a social preference function $F(\boldsymbol{\pi}, Z)$ has a property of:

Pareto efficiency (PE) if whenever alternative a is ranked above b according to each π_i, then a is ranked above b according to $F(\boldsymbol{\pi}, Z)$.

Independency of irrelevant alternatives (IIA) if whenever the ranking of a versus b is unchanged for each i = 1, 2, …, n when individual i's ranking changes from πi to $\pi' i$; then the ranking of a versus b is the same according to both $F(\boldsymbol{\pi}, Z)$ and $F(\boldsymbol{\pi}', Z)$

Dictatorship (D) if there is an individual i such that $F(\boldsymbol{\pi}, Z) = \pi_i$ (one alternative is ranked above another in the social ranking whenever the one is ranked above the other according to the individual ranking π_i).

Strategic voting manipulability if there exists a preference profile $\boldsymbol{\pi}$, a subset of individuals $K \subset N$ and a preference profile $\boldsymbol{\pi}'$ such that $\pi'_i = \pi_i$ for $i \in N \backslash K$, $F(\boldsymbol{\pi}, Z) = \pi_0$, $F(\boldsymbol{\pi}', Z) = \pi'_0$, and for all $i \in K$ it holds that $d(\pi_i, \pi_0) < d(\pi_i, \pi'_0)$.

Strategic nomination manipulability in Z if there exist Z' such that $Z' \subset Z$, a subset of individuals $M \subset N$ and preference profiles $\boldsymbol{\pi}$ on Z and $\boldsymbol{\pi}'$ on Z' where $\boldsymbol{\pi}'$ is a truncated preference profile $\boldsymbol{\pi}$ with the same individual preferences for Z as for Z', such that $F(Z, \boldsymbol{\pi}) = \pi_0 \in \Pi^*(Z)$, $F(Z', \boldsymbol{\pi}') = \pi'_0 \in \Pi^*(Z')$ and for all $i \in N \backslash M$ it holds that $d(\pi_i(Z'), \pi_0) < d(\pi_i(Z), \pi'_0)$.

2.5 Examples of manipulation

To illustrate concepts of strategic voting and strategic nomination we shall use the Borda social choice function and Borda social preference function.

Let $N(x,y, \boldsymbol{\pi})$ be number of voters who prefer x to y ($x, y \in Z$), given a preference profile $\boldsymbol{\pi}$. Function

$$\phi(x,\boldsymbol{\pi}) = \sum_{y \in Z} N(x,y,\boldsymbol{\pi})$$

measures how many times a candidate x was preferred to the other candidates y for all $y \in Z$.

Borda's social choice function

$$f(Z,\boldsymbol{\pi}) = \{x : x = \arg\max_{z \in Z} \phi(z,\boldsymbol{\pi})\}$$

chooses the candidate that received the maximum total number of votes in all pair-wise comparisons to other candidates. Borda's social preference function ranks the alternatives in order of the values of the function $\phi(x, \boldsymbol{\pi})$.

Example 2.5: strategic voting.

Consider three alternatives {A, B, C} and 90 voters divided into four groups with identical preferences of each group: (1) of 20 voters, (2) of 20 voters, (3) of 20 voters, and (4) of 30 voters. In Table 1a we provide preference profile $\boldsymbol{\pi} = (\pi_{1,} \pi_{2,} \pi_3, \pi_3)$:

Table 1a

(1)	(2)	(3)	(4)
π_1	π_2	π_3	π_4
20	20	20	30
A	A	C	B
B	C	B	A
C	B	A	C

In Table 1b see the matrix of pair-wise comparisons for preference profile $\boldsymbol{\pi}$.

Table 1b

	A	B	C	
A	0	40	70	110
B	50	0	50	100
C	20	40	0	60

Assuming sincere voting the Borda winner is A, Borda social ranking [A, B, C].

If the group (4) of 30 voters with honest orderings π_4 decides to misrepresent their true preferences by π'_4 and the other voters are following their true preferences, we move to the preference profile $\boldsymbol{\pi} = (\pi_{1,} \pi_{2,} \pi_3, \pi'_4)$, see Table 2a:

Table 2a

(1)	(2)	(3)	(4)
π_1	π_2	π_3	π'_4
20	20	20	30
A	A	C	B
B	C	B	C
C	B	A	A

The matrix of pair-wise comparisons (Table 2b):

Table 2b

	A	B	C	
A	0	40	40	80
B	50	0	50	100
C	40	40	0	80

The Borda winner is B, the Borda social ranking [B, (A,C)]. There exists an incentive for strategic voting of the group (3).

Example 2.6: strategic nomination.

Consider three alternatives Z = {A, B, C} and 79 voters divided into three groups with identical preferences of each group: (1) of 20 voters, (2) of 24 voters, and (3) of 35 voters. In Table 3a we provide preference profile $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$:

Table 3a

(1)	(2)	(3)
π_1	π_2	π_3
20	24	35
A	B	C
B	C	A
C	A	B

In Table 3b see the matrix of pair-wise comparisons for preference profile $\boldsymbol{\pi}$.

Table 3b

	A	B	C	
A	0	55	20	75
B	55	0	44	99
C	59	35	0	94

Assuming sincere voting, the Borda winner is B, Borda social ranking [B, C, A].

Assume that there exists an alternative D and the preference profile $\boldsymbol{\pi'} = (\pi'_1, \pi'_2, \pi'_3)$ of voters' groups on the set of alternatives Z' = {A, B, C, D}, see Table 4b.

Table 4b

(1)	(2)	(3)
π'_1	π'_2	π'_3
20	24	35
A	B	C
D	C	A
B	A	D
C	D	B

The corresponding matrix of pair-wise comparisons (Table 4b):

Table 4b

	A	B	C	D	
A	0	55	20	79	154
B	55	0	44	24	123
C	59	35	0	59	153
D	0	55	20	0	75

The Borda winner is A, the Borda social ranking [A, C, B, D]. There exists an incentive for group (1) to nominate alternative D.

3. Dictatorship versus manipulability?

Two famous social choice theorems are related to the problems of dictatorship and manipulability. While the Arrow's "impossibility" theorem is usually associated with non-existence of non dictatorial social preference function, the Gibbard-Satterthwaite theorem shows that any non-dictatorial non-degenerate social choice function is manipulable. In fact, many authors observe that the both theorems are closely related (Reny, 2000). In this part of the paper we try to reformulate Arrow's and Gibbard-Satterthwaite theorems in terms of manipulability.

Gibbard-Satterthwaite theorem 1:

If card (Z) $\geq$ 3, and social choice function f(Z, π) satisfies Pareto efficiency, non-dictatorship and non-degeneracy, then f(Z, π) is manipulable.

Gibbard-Satterthwaite theorem 2:

If card (Z) $\geq$ 3, and social choice function f(Z, π) satisfies Pareto efficiency, monotonicity and non-degeneracy, then f(Z, π) is dictatorial.

Arrow theorem 1

If card (Z) ≥ 3, and the social preference function F(Z, π) satisfies Pareto efficiency and non-dictatorship, then F(Z, π) is manipulable.

Arrow theorem 2

If card (Z) ≥ 3, and social preference function F(Z, π) satisfies Pareto efficiency and independence of irrelevant alternatives, then F(Z, π) is dictatorial.

Monotonicity is a special case of independency of irrelevant alternatives. A social choice function is non manipulable if and only if it is monotonic. A social preference function is not manipulable if and only if it satisfies the independence of irrelevant alternatives. The conjecture is that Gibbard-Satterthwaite theorem is a special case of Arrow.

4. Concluding remarks

Since the Arrow's result was first published in 1951, a vast literature has grown on impossibility theorem. The great debate started about practical political conclusions from the Arrow's result. In the same way, the Gibbard-Satherthwaite theorem raised questions about how people will behave in making social decisions. For example: what sorts of strategies will they adopt when they are all voting dishonestly? What is the equilibrium when everybody is "cheating"? Theorems imply the problem of political legitimacy: in a world in which voters are misrepresenting their preferences, it is difficult to say that the outcome selected is "right", "correct" or "legitimate". Suppose for instance that candidate A wins an election process in which there were several other candidates, and the people "slightly misrepresented" their "true" preferences. Is the candidate A in such case a legitimate people's choice?

The question is: why so strictly insists on "non-manipulability"? Voting is a game, with, perhaps, imperfect information. The outcome depends on choices made by many independent decision makers. Strategic rationality of voters is a standard assumption in theory of decision. Any manipulable social choice function is better than dictatorship.

While the great achievement of Arrow and Gibbard-Satterthwaite impossibility theorem was to state the problem and to show that this sort of problems can be analyzed in a general framework of the application of rigorous mathematical methods to the social sciences, there is no reason for resigning on analyzing of particular social choice procedures and considering all of them equally bad or unusable.

Problems to be addressed:

a) Is Gibbard-Sattethwaite a special case of Arrow (a rigorous proof)?
b) Different types of voters' rationality.
c) Different types of manipulation.
d) Information complexity of manipulation.
e) Voting as a game: models of multilateral manipulation.

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