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Isoperimetric Problem in Economics

Bakalářská práce

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Abstrakt

Isoperimetrická úloha patří do širokého okruhu úloh teorie optimálního řízení, jež se vyvinula v polovině 20. století jako odvětví variačního počtu. V teoretické části této bakalářské práce je uvedena přesná formulace úlohy a proveden důkaz jak nutné, tak postačující podmínky pro existenci řešení.

V další kapitole je představena jednoduchá úloha týkající se odvětví aplikované ekonomie – alokace zdrojů za účelem minimalizace nákladů. Na základě článku z Management Science (Cullingford a Prideaux, 1973) představuji účelovou funkci s přidaným diskontním faktorem. Je uvedeno řešení úlohy a jeho grafické znázornění.

V poslední části jsou představeny možnosti rozšíření úlohy o jednu či více dodatečných omezujících podmínek. Tyto podmínky mohou mít tvar rovnosti či nerovnosti.

Klíčová slova

variační počet, teorie optimálního řízení, plánování, náklady na řízení zdrojů, diskont, neaktivní omezující podmínky

Abstract

The isoperimetric problem is one of the broad class of optimal control problems, which draw on the generalization of classical calculus developed in the mid-20th century. In the bachelor's thesis I lay down the mathematical framework that permits to rigorously prove both the necessary and sufficient conditions for the existence of a maximizer of the objective function.

I analyze a simple problem from the field of project planning, which is a branch of applied economics. On the basis of a 1973 article by Cullingford and Prideaux I present an augmented cost function that involves the concept of the time value of money, which is key to proper economic reasoning. I give an explicit solution along with graphical depictions of the impact of a non-zero discount factor on project planing under the model in question.

Finally, I introduce additional constraints and discuss the subproblem of multiple equality and non-equality constraints.

Keywords

calculus of variations, optimal control, project planning, resource variation cost, discounting, inactive constraints

Prohlášení

1. Prohlašuji, že jsem předkládanou práci zpracoval samostatně a použil jen uvedené prameny a literaturu. Prohlašuji, že práce nebyla využita k získání jiného nebo stejného titulu.

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Poděkování

Děkuji svému konzultantovi Doc. RNDr. Oldřichu Johnovi, CSc., za obětavý přístup a pečlivé vedení.

M. V.

(Teze Bc. práce)

Contents

1 Introduction

1.1 Overview of the isoperimetric problem

The calculus of variations, which by the 20th century had become an established branch of modern mathematics, found its way to the focus of economists in the 1950s and 1960s when the Optimal control theory was developed. The work of L. S. Pontryagin, R. Bellman and others has since been built on by economists, management scientists and the like (Kamien and Schwartz, 1991, pp. 3–4).

A typical problem in the calculus of variations seeks to find a function which is a stationary point (e.g., a maximum point) of a functional, where the latter is typically in the form of a definite integral (Weisstein, 2010b). In optimal control theory, a pair or, more generally, a vector of functions is sought which maximizes a given functional.

Constraints are additional requirements we impose on the solution, thus narrowing the set of candidate solutions—the feasible set. Where the constraint is in the form of a definite integral, we speak of an *isoperimetric constraint*. The general class of problems in the calculus of variations that involve this type of constraints are called *isoperimetric problems*. This bachelor's thesis deals with a particular formulation of the isoperimetric problem, given in detail in Chapter 2.

1.2 Literature and applications

A broad overview of optimal control problems is found in Bryson and Ho (1975, pp. 90–95) where isoperimetric problems are referred to as problems with "integral constraints". Berkovitz (1974, pp. 30 and 190) lists the isoperimetric problem as a possible formulation of the optimal control problem and gives its solution using the more general Pontryagin maximum principle. Kamien and Schwartz (1991, pp. 47–51) treat the isoperimetric problem in a section of its own right and state the necessary condition for the existence of a solution.

Cullingford and Prideaux (1973) present an example of how the isoperimetric problem might be applied to project planning. They seek to minimize the cost associated with the variation of a resource that is allocated to a task subject to the constraint that defines the total amount of resource used. The problem is developed in this bachelor's thesis (Chapter 3).

The isoperimetric problem can be applied to resource management where optimal path of extraction of an exhaustible resource is sought subject to a quantity constraint (Bobková, 2009).

Anjos et al. (2005) apply the isoperimetric problem to yield management. They seek to maximize revenue from the sale of airline tickets subject to a capacity constraint. Their method, which appears to have been adopted by an actual airline, is tested in Anjos et al. (2004).

1.3 Origin of the name

The isoperimetric problem draws its name from an ancient mathematical problem that traces back to Zenodorus (Wiegert, 2010). The problem asks to find a planar figure of a given perimeter that encloses the maximum possible area. Although the circle seemed to the ancient Greeks to be the obvious solution, this was not rigorously proved until 1841 by the Swiss mathematician Jakob Steiner (Weisstein, 2010a).

The "original" isoperimetric problem has been studied extensively since then, giving rise to various generalizations. For instance, the isoperimetric inequality on various manifolds is treated thoroughly in Chavel (2001), Harper (2004) deals with variants of the isoperimetric problem in graph theory, called edge-isoperimetric and vertex-isoperimetric problems, etc.

1.4 Structure of the work

This bachelor's thesis is structured as follows. First come the introductory paragraphs on the background of the isoperimetric problem. Then, in Chapter 2, the problem is formulated along with all the necessary definitions. A necessary condition for the existence of a solution is given along with a detailed proof. The same is done for the sufficient condition for existence, which relies on a more stringent type of isoperimetric constraint.

In Section 3.1 the problem from Cullingford and Prideaux (1973) is analyzed. Section 3.2 presents a modification of the objective function by the introduction of a discount factor $e^{-\rho t}$. This approach seems, in some cases, to correspond better to real-world situations. In Section 3.3 I introduce additional constraints. This is done in three ways. First, one integral constraint is introduced and an explicit formula for the solution is given. Second, a possible formulation of the problem involving more such constraints is given. Third, the implications regarding inequality constraints are discussed.

2 Isoperimetric problem

2.1 Preliminaries and notations

This chapter gives insight into what Hadley and Kemp (1971, p. 22) call "the simplest problem in the calculus of variations". Let us first extend the notion of a continuously differentiable function.

Definition Let $a, b \in \mathbb{R}, a < b$. We say that a function $x : [a, b] \to \mathbb{R}$ is of class C^1 on $[a, b]$, or $x \in C^1[a, b]$, if the function

$$\dot{x}(t) = \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t}(a_+) & \text{for } t = a \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t) & \text{for } t \in (a, b) \\ \frac{\mathrm{d}x}{\mathrm{d}t}(b_-) & \text{for } t = b \end{cases}$$

exists and is continuous on $[a, b]$.[1]

Definition By $C^1([0, T] \times \mathbb{R}^n), n \in \mathbb{N}$, we denote the class of real-valued functions of $n + 1$ variables defined on $[0, T] \times \mathbb{R}^n$, with all their partial derivatives of first order continuous on $(0, T) \times \mathbb{R}^n$, which can be continuously extended on $[0, T] \times \mathbb{R}^n$.

Definition Let $A, B, G_0, T \in \mathbb{R}, T > 0$ and let $f, g \in C^1([0, T] \times \mathbb{R}^2)$. Define functionals $F, G : C^1[0, T] \to \mathbb{R}$ and set M as

$$F(y) = \int_0^T f(t, y(t), \dot{y}(t))\, \mathrm{d}t, \tag{1}$$

$$G(y) = \int_0^T g(t, y(t), \dot{y}(t))\, \mathrm{d}t, \tag{2}$$

$$M = \left\{ y \mid y \in C^1[0, T],\ y(0) = A,\ y(T) = B,\ G(y) = G_0 \right\}. \tag{3}$$

[1]Briefly on properties of the set $C^1[a, b]$. First, it is a vector space over $\mathbb{R}$. Second, all functions in $C^1[a, b]$ are continuous on $[a, b]$, a fact that may be written as $C^1[a, b] \subset C[a, b]$. Third, if J is an interval such that $[a, b] \subset J$ then the restriction of any function $f \in C^1(J)$ to $[a, b]$ is of class C^1 on $[a, b]$.

We consider the following problem. Find $y_0 \in M$ such that

$$\forall y \in M \quad F(y) \leq F(y_0). \tag{4}$$

The functional F is called the *objective function*, the constraint $G(y) = G_0$ the *isoperimetric constraint* and the constraints $y(0) = A$, $y(T) = B$ the *terminal* or *boundary conditions*. The set M of functions that satisfy all constraints (*candidate solutions*) is called the *feasible set*. The function y_0 is called the *maximizer of functional F on the set M*.

In Sections 2.2 and 2.3 we derive necessary and sufficient conditions for the existence of a maximizer of F on M.

Remark Let $i, n \in \mathbb{N}, i \leq n$. The symbol $\partial_i k$ shall denote the partial derivative of the function $k : \mathbb{R}^n \to \mathbb{R}$ with respect to the i^{th} variable, that is

$$\partial_i k : [x_1, x_2, \ldots, x_n] \to \frac{\partial k}{\partial x_i}(x_1, x_2, \ldots, x_n). \tag{5}$$

In cases where a distinction between a vector and a scalar should be made the symbols printed in boldface denote vectors and vector-valued functions, e.g., $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ for $\mathbf{a} \in \mathbb{R}^n$.

2.2 Necessary condition

Definition Denote $X = \{h \mid h \in C^1[0,T], h(0) = h(T) = 0\}$ a vector subspace of $C^1[0,T]$.

Definition Let V be a vector space over $\mathbb{R}$ and $a, h \in V$. Let $K : V \to \mathbb{R}$. The *directional derivative* of K at point a in direction h is defined as

$$\delta K(a, h) = \lim_{\tau \to 0} \frac{K(a + \tau h) - K(a)}{\tau} \tag{6}$$

if the limit exists and is finite.

Lemma 1 Let x be a non-negative continuous function on $[0,T]$ for which $\int_0^T x\left(t\right)\,\mathrm{d}t = 0$. Then $x\left(t\right)=0$ for all $t\in[0,T]$.

Proof Let there be a $t\in(0,T)$ satisfying $x\left(t\right)>0$. Due to the continuity of x at point t there exists $\delta>0$ such that for all $v\in(t-\delta,t+\delta)\cap[0,T]: x\left(v\right)>\frac{1}{2}x\left(t\right)$. But $\int_0^T x\,\mathrm{d}v=\int_0^{t-\delta}x\,\mathrm{d}v+\int_{t-\delta}^{t+\delta}x\,\mathrm{d}v+\int_{t+\delta}^{T}x\,\mathrm{d}v\geq 0+2\delta\frac{1}{2}x\left(t\right)+0=\delta x\left(t\right)>0$ which contradicts the assumption that the integral be zero. For $x\left(0\right)>0$ or $x\left(T\right)>0$ we can proceed analogically. Therefore x must be identically zero on $[0,T]$, QED.

Lemma 2 (Du Bois–Reymond Lemma) Let a,b be continuous functions on $[0,T]$. Then

$$\forall h\in X\quad\int_0^T\left(a\left(t\right)h\left(t\right)+b\left(t\right)\dot{h}\left(t\right)\right)\,\mathrm{d}t=0 \tag{7}$$

if and only if $\dot{b}$ exists and $\dot{b}\left(t\right)=a\left(t\right)$ for all $t\in[0,T]$.

Proof Let (7) hold. Since a is continuous on $[0,T]$ it has a Riemann integral on $[0,T]$. Let A be one of the antiderivatives. Define $w:t\to\alpha t+\int_0^t\left(b\left(u\right)-A\left(u\right)\right)\,\mathrm{d}u$ where $\alpha\in\mathbb{R}$ is chosen so that $w\left(T\right)=0$. Then obviously $w\in X$ and we can write

$$0=\int_0^T\left(aw+b\dot{w}\right)\,\mathrm{d}t=\left[Aw\right]_0^T+\int_0^T\left(-A\dot{w}+b\dot{w}\right)\,\mathrm{d}t= \tag{8}$$

$$=\int_0^T\dot{w}\left(b-A\right)\,\mathrm{d}t=\int_0^T\dot{w}\left(b-A+\alpha\right)\,\mathrm{d}t= \tag{9}$$

$$=\int_0^T\left(b-A+\alpha\right)\left(b-A+\alpha\right)\,\mathrm{d}t=\int_0^T\left(b-A+\alpha\right)^2\,\mathrm{d}t. \tag{10}$$

In (8) we integrate aw by parts. In (9) we use the fact that $A\left(T\right)w\left(T\right)=A\left(0\right)w\left(0\right)=0$ and that $\int_0^T\alpha\dot{w}\,\mathrm{d}t=\alpha w\left(T\right)-\alpha w\left(0\right)=0$. In (10) we substitute for $\dot{w}$.

Finally, application of Lemma 1 to the function $x=\left(b-A+\alpha\right)^2$ which is obviously non-negative and continuous on $[0,T]$ yields $\left(b-A+\alpha\right)^2=0$, or $b=A-\alpha$, for all $t\in[0,T]$. Thus, on $[0,T]$, b is identical to $A-\alpha$, the derivative of which exists and is equal to a.

Conversely, suppose that $\dot{b}$ exists and $\dot{b}(t) = a(t)$ for all $t \in [0,T]$. Then

$$\int_0^T \left(a(t)\,h(t) + b(t)\,\dot{h}(t)\right)\,\mathrm{d}t = b(T)\,h(T) - b(0)\,h(0) = 0$$

for all $h \in X$, QED.

Definition Let $k \in C^1([0,T] \times \mathbb{R}^2)$, $y \in C^1[0,T]$ and let $\frac{\mathrm{d}}{\mathrm{d}t}\partial_3 k(t, y(t), \dot{y}(t))$ exist[2] and be finite on $[0,T]$. We say that the pair (k,y) *satisfies the Euler–Lagrange equation* if

$$\forall t \in [0,T] \quad \partial_2 k(t, y(t), \dot{y}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\partial_3 k(t, y(t), \dot{y}(t)) = 0.$$

Lemma 3 Let $k \in C^1([0,T] \times \mathbb{R}^2)$, $y \in C^1[0,T]$ and let $K : C^1[0,T] \to \mathbb{R}$ be defined as $K(y) = \int_0^T k(t, y(t), \dot{y}(t))\,\mathrm{d}t$. Then (k,y) satisfies the Euler–Lagrange equation if and only if

$$\forall h \in X \quad \delta K(y,h) = 0. \tag{11}$$

Proof Let us write, using (6),

$$\delta K(y,h) = \lim_{\tau \to 0} \frac{K(y + \tau h) - K(y)}{\tau} = \lim_{\tau \to 0} \int_0^T \psi(t,\tau)\,\mathrm{d}t$$

where ψ is defined as

$$\psi(t,\tau) = \begin{cases} \frac{1}{\tau}\left(k\left(t, y(t) + \tau h(t), \dot{y}(t) + \tau \dot{h}(t)\right) - k(t, y(t), \dot{y}(t))\right) & \text{for } \tau \neq 0, \\ h(t)\,\partial_2 k(t, y(t), \dot{y}(t)) + \dot{h}(t)\,\partial_3 k(t, y(t), \dot{y}(t)) & \text{for } \tau = 0. \end{cases}$$

Due to both y and k being continuously differentiable, the limit $\lim_{\tau \to 0} \psi(t,\tau)$ exists and is finite for every t. From the Chain Rule it follows that it is equal to $\psi(t,0)$. Therefore ψ is continuous on $[0,T] \times \mathbb{R}$ and, according to Rektorys (2000,p. 510),

$$\lim_{\tau \to 0} \int_0^T \psi(t,\tau)\,\mathrm{d}t = \int_0^T \psi(t,0)\,\mathrm{d}t.$$

[2]in the sense that the function $t \to \partial_3 k(t, y(t), \dot{y}(t))$ is of class C^1 on $[0,T]$

Thus

$$\delta K(y,h) = \int_0^T \Big(h(t)\,\partial_2 k(t, y(t), \dot{y}(t)) + \dot{h}(t)\,\partial_3 k(t, y(t), \dot{y}(t)) \Big)\,\mathrm{d}t. \qquad (12)$$

Define $a : t \to \partial_2 k(t, y(t), \dot{y}(t))$, $b : t \to \partial_3 k(t, y(t), \dot{y}(t))$. As an easy consequence of the continuity of $\partial_2 k$ on $[0,T]\times\mathbb{R}^2$ and the continuity of y and $\dot{y}$ on $[0,T]$ we obtain the continuity of both a, b on $[0,T]$. According to (12) and Lemma 2, statement (11) holds if and only if b is of class C^1 on $[0,T]$ and

$$\forall t \in [0,T] \quad \partial_2 k(t, y(t), \dot{y}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\partial_3 k(t, y(t), \dot{y}(t)), \qquad (13)$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ stands for respective one-sided derivatives for $t \in \{0, T\}$. But this is equivalent to the statement that (k, y) satisfies the Euler–Lagrange equation, QED.

Theorem 4 (Implicit function theorem) Let $\mathbf{z} \in \mathbb{R}^4$, $\boldsymbol{\varphi} : \mathbb{R}^4 \to \mathbb{R}^2$, $\varphi_1, \varphi_2 \in C^1(\mathbb{R}^4)$, $\boldsymbol{\varphi}(\mathbf{z}) = \mathbf{0}$ and let the matrix

$$\begin{bmatrix} \partial_3\varphi_1(\mathbf{z}) & \partial_4\varphi_1(\mathbf{z}) \\ \partial_3\varphi_2(\mathbf{z}) & \partial_4\varphi_2(\mathbf{z}) \end{bmatrix}$$

be nonsingular. Then there exist neighborhoods $U \subset \mathbb{R}^2$ of $[z_1, z_2]$ and $V \subset \mathbb{R}^2$ of $[z_3, z_4]$ such that for each $\mathbf{u} \in U$ there exists a unique $\mathbf{v} \in V$ which satisfies $\boldsymbol{\varphi}(u_1, u_2, v_1, v_2) = \mathbf{0}$.

Theorem 5 (Necessary condition) Let $y_0 \in M$ be a maximizer of F on M. Then either (g, y_0) satisfies the Euler–Lagrange equation or there exists $\lambda \in \mathbb{R}$ such that $(f + \lambda g, y_0)$ satisfies the Euler–Lagrange equation.

Proof The proof follows the idea of Alexejev et al. (1991, pp. 66–69). Suppose that y_0 is a maximizer of F on M and (g, y_0) does not satisfy the Euler–Lagrange equation. By Lemma 3 there exists $y \in X$ such that $\delta G(y_0, y) \neq 0$. Let x be any function from X and let us define $\boldsymbol{\varphi}^{x,y} : \mathbb{R}^4 \to \mathbb{R}^2$ as

$$\boldsymbol{\varphi}^{x,y}(\mathbf{z}) = \begin{bmatrix} F\left(y_0 + z_3 x + z_4 y\right) - z_1 \\ G\left(y_0 + z_3 x + z_4 y\right) - z_2 \end{bmatrix}.$$

From the assumption that $f, g \in C^1\left([0,T] \times \mathbb{R}^2\right)$ it follows that $\varphi_1^{x,y}, \varphi_2^{x,y} \in C^1\left(\mathbb{R}^4\right)$. Let $\mathbf{z} = \left[F\left(y_0\right), G\left(y_0\right), 0, 0\right]$. Note that $\boldsymbol{\varphi}^{x,y}\left(\mathbf{z}\right) = \mathbf{0}$ and that there exists finite

$$\partial_3 \varphi_1^{x,y}\left(\mathbf{z}\right) = \lim_{\tau \to 0} \frac{\varphi_1^{x,y}\left(F\left(y_0\right), G\left(y_0\right), \tau, 0\right) - \varphi_1^{x,y}\left(\mathbf{z}\right)}{\tau} = \\ = \lim_{\tau \to 0} \frac{F\left(y_0 + \tau x\right) - F\left(y_0\right) - F\left(y_0\right) + F\left(y_0\right)}{\tau} = \delta F\left(y_0, x\right).$$

It is similarly shown that

$$\mathbb{D}_{x,y} = \begin{bmatrix} \partial_3 \varphi_1^{x,y}\left(\mathbf{z}\right) & \partial_4 \varphi_1^{x,y}\left(\mathbf{z}\right) \\ \partial_3 \varphi_2^{x,y}\left(\mathbf{z}\right) & \partial_4 \varphi_2^{x,y}\left(\mathbf{z}\right) \end{bmatrix} = \begin{bmatrix} \delta F\left(y_0, x\right) & \delta F\left(y_0, y\right) \\ \delta G\left(y_0, x\right) & \delta G\left(y_0, y\right) \end{bmatrix}. \tag{14}$$

Assume towards contradiction that $\mathbb{D}_{x,y}$ is nonsingular for some $x \in X$. Then all assumptions for Theorem 4 hold and for any sufficiently small $\epsilon > 0$ there exists $\mathbf{v} \in \mathbb{R}^2$ such that

$$\boldsymbol{\varphi}^{x,y}\left(F\left(y^0\right) + \epsilon, G\left(y^0\right), v_1, v_2\right) = \\ = \begin{bmatrix} F\left(y_0 + v_1 x + v_2 y\right) - F\left(y_0\right) - \epsilon \\ G\left(y_0 + v_1 x + v_2 y\right) - G\left(y_0\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{15}$$

Denote $y_\epsilon\left(t\right) = y_0\left(t\right) + v_1 x\left(t\right) + v_2 y\left(t\right), t \in [0,T]$. Obviously $y_\epsilon\left(0\right) = y_0\left(0\right) = A$, $y_\epsilon\left(T\right) = y_0\left(T\right) = B$. From (15) it follows that $G\left(y_\epsilon\right) = G\left(y_0\right) = G_0$, thus $y_\epsilon \in M$, but $F\left(y_\epsilon\right) = F\left(y_0\right) + \epsilon > F\left(y_0\right)$, so y_0 is not a maximizer of F on M. This is in contradiction with the assumptions and consequently for all $x \in X$ the matrix $\mathbb{D}_{x,y}$ must be singular. We can write

$$\forall x \in X \quad \frac{\det \mathbb{D}_{x,y}}{\delta G\left(y_0, y\right)} = \delta F\left(y_0, x\right) - \frac{\delta F\left(y_0, y\right)}{\delta G\left(y_0, y\right)} \delta G\left(y_0, x\right) = 0. \tag{16}$$

Finally, put

$$\lambda = -\frac{\delta F(y_0, y)}{\delta G(y_0, y)}.$$

From the fact that $x \in X$ has been chosen arbitrarily, from (6) and from (16) it follows immediately that

$$\forall x \in X \quad \delta(F + \lambda G)(y_0, x) = 0$$

which is, by Lemma 3, equivalent to the statement that $(f + \lambda g, y_0)$ satisfies the Euler–Lagrange equation, QED.

2.3 Sufficient condition

Under certain circumstances we are able to guarantee the existence of a solution of the isoperimetric problem.

Lemma 6 Let $g \in C^1([0,T] \times \mathbb{R}^2)$ be such that

$$g(t, a_1, b_1) + g(t, a_2, b_2) = g(t, a_1 + a_2, b_1 + b_2) \tag{17}$$

for all $t \in [0,T]$ and all $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then M is convex.

Proof From (17) and from the fact that g is continuous on $[0,T] \times \mathbb{R}^2$ it follows that

$$g(t, \eta a, \eta b) = \eta g(t, a, b) \tag{18}$$

for all $t \in [0,T]$ and all $\eta, a, b \in \mathbb{R}$. Let $y_1, y_2 \in M$, $\tau \in [0,1]$ and let $y = \tau y_1 + (1-\tau) y_2$ on $[0,T]$. It is easily verified that $y \in C^1[0,T], y(0) = A, y(T) = B$. Furthermore, using (17) and (18),

$$\begin{aligned} G(y) &= \int_0^T g(t, \tau y_1(t) + (1-\tau) y_2(t), \tau \dot{y}_1(t) + (1-\tau) \dot{y}_2(t))\, \mathrm{d}t = \\ &= \int_0^T (\tau g(t, y_1(t), \dot{y}_1(t)) + (1-\tau) g(t, y_2(t), \dot{y}_2(t)))\, \mathrm{d}t = \\ &= \tau G(y_1) + (1-\tau) G(y_2) = G_0. \end{aligned} \tag{19}$$

Thus $y \in M$, QED.

Definition Let V be a vector space over $\mathbb{R}$. Let $K : V \to \mathbb{R}$. Let S be a convex subset of V. The functional K is *concave on* S if for all $x_1, x_2 \in S$ and all $\tau \in [0, 1]$

$$K(\tau x_1 + (1-\tau) x_2) \geq \tau K(x_1) + (1-\tau) K(x_2) \tag{20}$$

Lemma 7 Let the bivariate functions $f_t : \mathbf{u} \to f(t, u_1, u_2)$ be concave on $\mathbb{R}^2$ for all $t \in [0, T]$. Then F is concave on $C^1[0, T]$.

Proof Assuming the concavity of f_t's and using (20) we can write

$$\begin{aligned}
&\forall t \in [0,T], \forall y_1, y_2 \in C^1[0,T], \forall \tau \in [0,1] : \\
&\qquad f_t(\tau y_1(t) + (1-\tau) y_2(t), \tau \dot{y}_1(t) + (1-\tau)\dot{y}_2(t)) \geq \\
&\qquad\qquad \geq \tau f_t(y_1(t), \dot{y}_1(t)) + (1-\tau) f_t(y_2(t), \dot{y}_2(t))
\end{aligned}$$

Therefore, for all $y_1, y_2 \in C^1[0, T]$ and all $\tau \in [0, 1]$,

$$\begin{aligned}
&\int_0^T f(t, \tau y_1(t) + (1-\tau) y_2(t), \tau \dot{y}_1(t) + (1-\tau)\dot{y}_2(t))\, \mathrm{d}t \geq \\
&\qquad \geq \tau \int_0^T f(t, y_1(t), \dot{y}_1(t))\, \mathrm{d}t + (1-\tau) \int_0^T f(t, y_2(t), \dot{y}_2(t))\, \mathrm{d}t
\end{aligned}$$

which is precisely the statement of concavity of F on $C^1[0, T]$, QED.

Theorem 8 (Sufficient condition) Let g be as in Lemma 6 and let F be concave on M, $y_0 \in M$ and let $\lambda \in \mathbb{R}$ be such that $(f + \lambda g, y_0)$ satisfies the Euler–Lagrange equation. Then y_0 is a maximizer of F on M.

Proof Define $L = F + \lambda G$ on $C^1[0, T]$. By Lemma 6 M is convex. Assuming the concavity of F on M and using (19) we can write, for all $z \in M$ and all $\tau \in (0,1]$,

$$\begin{aligned} F(\tau z + (1-\tau) y_0) + \lambda G_0 &\geq \tau F(z) + (1-\tau) F(y_0) + \lambda G_0 \\ F(y_0 + \tau(z - y_0)) + \lambda G(y_0 + \tau(z - y_0)) &\geq F(y_0) + \lambda G(y_0) + \tau(F(z) - F(y_0)) \\ \frac{L(y_0 + \tau(z - y_0)) - L(y_0)}{\tau} &\geq F(z) - F(y_0). \end{aligned}$$

Now $z - y_0 \in X$ for all $z \in M$ and by Lemma 3 $\delta L(y_0, h) = 0$ for all $h \in X$. Thus for all $z \in M$

$$0 = \delta L(y_0, z - y_0) = \lim_{\tau \to 0+} \frac{L(y_0 + \tau(z - y_0)) - L(y_0)}{\tau} \geq F(z) - F(y_0),$$

implying that y_0 is a maximizer of F on M, QED. (Both the limit and the directional derivative exist by assumption.)

3 Resource variation cost

3.1 Basic problem

In their brief article dealing with project planning, Cullingford and Prideaux (1973) present a simple application of calculus of variations. In this section we repeat shortly the main idea of the paper.

Let $W \in \mathbb{R}$ and let other notation be the same as in Chapter 2. The problem is to find a maximizer y_0 of the functional

$$F_0(y) = \int_0^T -\dot{y}^2(t)\, \mathrm{d}t \tag{21}$$

subject to

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 0 \\ \int_0^T y(t)\, \mathrm{d}t &= W \end{aligned}$$

In this mathematical formulation y stands for the volume of a resource (e.g., labour) allocated to a task at a particular point of time t. The task must be finished by time T and its size, measured by the total resource used (e.g., man-hours) is defined to be W—this is the isoperimetric constraint. The functional $(-F_0)$ is called the *resource variation cost function* and the problem that of *minimization of the resource variation cost.* The maximizer y_0 is called the *optimal resource profile.*

Cullingford and Prideaux (1973) also address a number of other problems such as the finding of an optimal project duration, T. As much as such problems form standard part of the calculus of variations, they are not discussed in this paper.

Proposition 9 Problem (21) has a unique solution

$$y_0(t) = 6\frac{W}{T^3} t (T - t). \tag{22}$$

on $[0,T]$.

Proof In terms of the notation from Section 2 we have $F = F_0$, $f(t,y,\dot{y}) = -\dot{y}^2$, $g(t,y,\dot{y}) = y$, $G_0 = W$ and $A = B = 0$. The feasible set is therefore $M = \left\{ y \,\middle|\, y \in C^1[0,T], y(0) = y(T) = 0, \int_0^T y(t)\,\mathrm{d}t = W \right\}$

From Lemma 6 it follows that M is convex and from Lemma 7 it follows that F is concave on M. We intend to use Theorem 8. Let $y_0 \in C^1[0,T]$ and $\lambda \in \mathbb{R}$. The pair $(f + \lambda g, y_0)$ satisfies the Euler–Lagrange equation if and only if $\dot{y}_0 \in C^1[0,T]$ and

$$\lambda - 2\ddot{y}_0 = 0 \tag{23}$$

on $[0,T]$. Equation (23) has trivial solution—it is satisfied if and only if

$$\forall t \in [0,T] \quad y_0(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3 \tag{24}$$

for some $\boldsymbol{\alpha} \in \mathbb{R}^3$.[3] The function y_0 belongs to M if the boundary conditions are satisfied, that is

$$\begin{bmatrix} y_0(0) \\ y_0(T) \\ G(y_0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ T^2 & T & 1 \\ \frac{1}{3}T^3 & \frac{1}{2}T^2 & T \end{bmatrix} \boldsymbol{\alpha} = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}. \tag{25}$$

For every $T > 0$ Equation (25) has a unique solution

$$\boldsymbol{\alpha} = \begin{bmatrix} -6\frac{W}{T^3} \\ 6\frac{W}{T^2} \\ 0 \end{bmatrix}. \tag{26}$$

Thus there exist $y_0 \in M$, $y_0(t) = -6\frac{W}{T^3}t^2 + 6\frac{W}{T^2}t$, and $\lambda = 4\alpha_1 = 24\frac{W}{T^3}$ such that the pair $(f + \lambda g, y_0)$ satisfies the Euler–Lagrange equation (23). The application of Theorem 8 finishes the proof.

[3] We should require $\alpha_1 = \frac{\lambda}{4}$ but since the choice of λ is arbitrary there is no loss of generality.

3.2 Introducing the discount factor

The approach of Cullingford and Prideaux (1973) as it was summarized in Section 3.1 represents a useful application of the isoperimetric problem in project planning. For such application to be of interest, however, it is crucial to take into account the concept of time value of money. The most convenient way to do so is to introduce continuous compounding with a positive discount factor ρ, resulting in a multiplicative factor $\mathrm{e}^{-\rho t}$ being appended to the formula for f.

Let us consider the following problem. Let $W, \rho \in \mathbb{R}$, $\rho > 0$ and let other notation be the same as in Chapter 2. Find $y_\rho \in C^1[0,T]$ which maximizes the functional

$$F_\rho(y) = \int_0^T -\mathrm{e}^{-\rho t}\dot{y}^2(t)\ \mathrm{d}t \tag{27}$$

subject to

$$y(0) = y(T) = 0$$
$$\int_0^T y(t)\ \mathrm{d}t = W$$

Here, the functional $(-F_\rho)$ is called the *discounted resource variation cost function* and the problem that of *minimization of the discounted resource variation cost.* The maximizer y_ρ can also be called the *optimal resource profile.*

Proposition 10 Problem (27) has a unique solution

$$y_\rho(t) = W\rho^2 \frac{\left(\mathrm{e}^{\rho t}-1\right)T\mathrm{e}^{\rho T} - \left(\mathrm{e}^{\rho T}-1\right)t\mathrm{e}^{\rho t}}{\left(\mathrm{e}^{\rho T}-1\right)^2 - \rho^2T^2\mathrm{e}^{\rho T}} \tag{28}$$

Proof We use the same notation as before to write, for a given $\rho > 0$, $F = F_\rho$, $f(t,y,\dot{y}) = -\mathrm{e}^{-\rho t}\dot{y}^2$, $g(t,y,\dot{y}) = y$, $G_0 = W$ and $A = B = 0$. Note that $\mathrm{e}^{-\rho t} > 0$. The feasible set is the same as in Section 3.1, that is

$$M = \left\{ y \,\middle|\, y \in C^1[0,T], y(0) = y(T) = 0, \int_0^T y(t)\ \mathrm{d}t = W \right\}.$$

From Lemma 6 it follows that M is convex and from Lemma 7 it follows that F is concave on M. Again, we want to use Theorem 8. Let $y_\rho \in C^1[0,T]$ and $\lambda \in \mathbb{R}$. Then $(f+\lambda g, y_\rho)$ satisfies the Euler–Lagrange equation if and only if

$$\lambda \mathrm{e}^{\rho t} - 2\left(\ddot{y}_\rho - \rho \dot{y}_\rho\right) = 0 \tag{29}$$

on $[0,T]$. Equation (29) is a second-order linear ordinary differential equation which is easily solved by the substitution $z_\rho = \dot{y}_\rho$ and the use of integrating factor $\mathrm{e}^{-\rho t}$, yielding

$$\begin{aligned}
\mathrm{e}^{-\rho t}\dot{z}_\rho - \rho \mathrm{e}^{-\rho t} z_\rho - \tfrac{\lambda}{2} &= 0\\
\left[\mathrm{e}^{-\rho t} z_\rho\right]' &= \tfrac{\lambda}{2}\\
z_\rho &= \left(\tfrac{\lambda}{2}t + c_1\right)\mathrm{e}^{\rho t}\\
y_\rho &= \tfrac{\lambda}{2\rho} t\mathrm{e}^{\rho t} + \left(\tfrac{c_1}{\rho} - \tfrac{\lambda}{2\rho^2}\right)\mathrm{e}^{\rho t} + c_2,
\end{aligned} \tag{30}$$

on $[0,T]$, where $c_1, c_2 \in \mathbb{R}$ are constants. Let $\boldsymbol{\beta} \in \mathbb{R}^3$ and put $\lambda = 2\rho\beta_1$, $c_1 = \beta_1 + \rho\beta_2$, $c_2 = \beta_3$. Then, without loss of generality, (30) becomes

$$y_\rho(t) = \beta_1 t\mathrm{e}^{\rho t} + \beta_2 \mathrm{e}^{\rho t} + \beta_3 \tag{31}$$

for $t \in [0,T]$. Formula (31) describes all functions $y_\rho \in C^1[0,T]$ for which there exists $\lambda \in \mathbb{R}$ such that $(f+\lambda g, y_\rho)$ satisfies the Euler–Lagrange equation (29). To determine which of these functions lie in M we formulate the boundary conditions and the isoperimetric constraint:

$$\begin{bmatrix} y_\rho(0)\\ y_\rho(T)\\ G(y_\rho)\end{bmatrix} = \begin{bmatrix} 0 & 1 & 1\\ T\mathrm{e}^{\rho T} & \mathrm{e}^{\rho T} & 1\\ \frac{\mathrm{e}^{\rho T}(\rho T - 1) + 1}{\rho^2} & \frac{\mathrm{e}^{\rho T} - 1}{\rho} & T\end{bmatrix}\boldsymbol{\beta} = \begin{bmatrix} 0\\ 0\\ W\end{bmatrix}. \tag{32}$$

For $T > 0$, $\rho > 0$ the matrix in (32) is nonsingular and the equation is satisfied by a unique vector

$$\boldsymbol{\beta} = \frac{W\rho^2}{\left(\mathrm{e}^{\rho T} - 1\right)^2 - \rho^2 T^2 \mathrm{e}^{\rho T}} \begin{bmatrix} 1 - \mathrm{e}^{\rho T} \\ T\mathrm{e}^{\rho T} \\ -T\mathrm{e}^{\rho T} \end{bmatrix} \tag{33}$$

Therefore there exists a unique function $y_\rho \in M$ and a unique number $\lambda \in \mathbb{R}$, $\lambda = 2\rho\beta_1$, such that $(f + \lambda g, y_\rho)$ satisfies the Euler–Lagrange equation. The application of Theorem 8 finishes the proof.

Figure (A) shows the function y_ρ for $T = 1$, $W = 1$ and several values of ρ. The case $\rho = 0$ is a quadratic function as shown in Section 3.1. For positive discount rates a smaller amount of resource is allocated in the beginning of the task.[4]

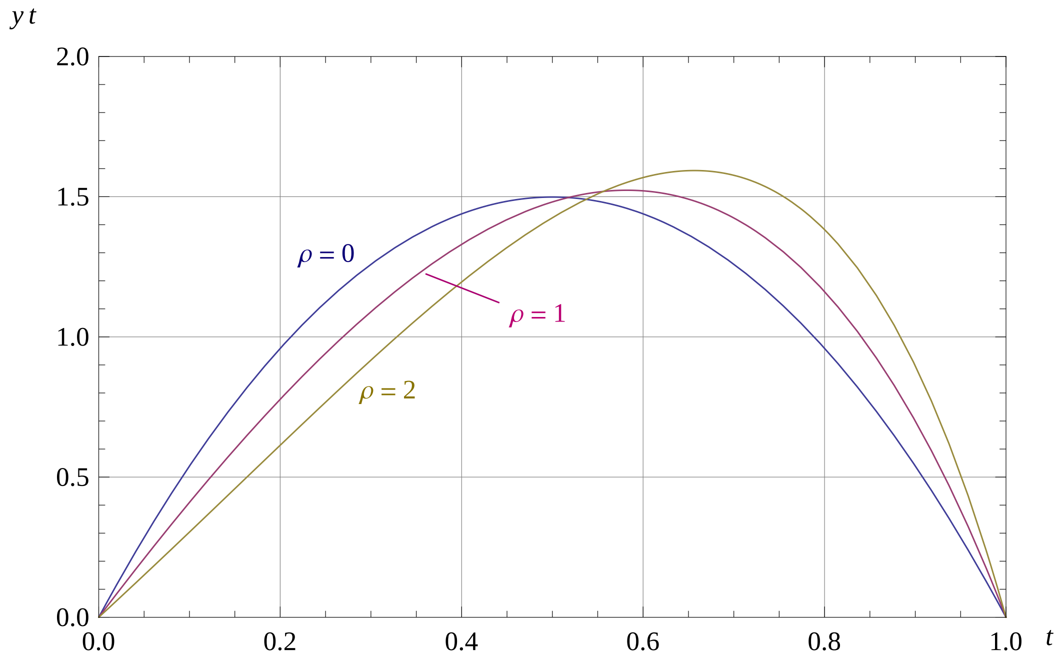

Figure (A): The optimal resource profile y_ρ under various discount rates.

The problem given in Cullingford and Prideaux (1973) is in fact a special case of the problem of minimization of the *discounted* cost function. In the Appendix (pages 32–37) we prove that their solutions (22) and (28), respectively, do not "differ

[4] All graphs were created using Mathematica 7, a software program.

much" from each other. Furthermore, we give a first-order approximation of y_ρ near $\rho = 0$. The same is done for the value of the objective function, $F_\rho(y_\rho)$.

Proposition 11 For each $t \in [0, T]$ there exists a function $\omega_t : \mathbb{R} \to \mathbb{R}$, $\lim_{\rho \to 0+} \omega_t(\rho) = 0$ and there exists a number $\rho_0 > 0$ such that for all $\rho \in [0, \rho_0)$

$$y_\rho(t) = \frac{2W}{T^3}(T - t)\, t\,(3 + \rho(2t - T)) + \rho\omega_t(\rho)$$

where y_ρ is defined in (22) and (28), respectively.

Proposition 12 There exists a function $\chi : \mathbb{R} \to \mathbb{R}$, $\lim_{\rho \to 0+} \chi(\rho) = 0$, and there exists a number $\rho_1 > 0$ such that for all $\rho \in [0, \rho_1)$

$$F_\rho(y_\rho) = -6\frac{W^2}{T^3}(2 - \rho T) + \rho\chi(\rho)$$

where F_ρ is defined in (21) and (27), respectively, and y_ρ is defined in (22) and (28), respectively.

3.2.1 Variable discount rate

The problem of minimization of the discounted resource variation cost can of course be further generalized *a*) by changing the way the future costs are discounted, or *b*) by changing the actual cost function. This can be summarized as follows. Let $W, T \in \mathbb{R}$, $T > 0$. Let $D \in C^1[0, T]$ be positive on $[0, T]$ and let $Q \in C^1(\mathbb{R})$ be strictly convex on $\mathbb{R}$. Find a maximizer $y \in C^1[0, T]$ of the functional

$$\tilde{F}(y) = \int_0^T -D(t)\, Q(\dot{y}(t))\, \mathrm{d}t \tag{34}$$

subject to $y(0) = A,\ y(T) = B,\ \int_0^T y(t)\, \mathrm{d}t = W$.

Proposition 13 If y is a solution it satisfies

$$y(t) = \int \dot{Q}^{-1}\left(\frac{\gamma_1 t + \gamma_2}{D(t)}\right) \mathrm{d}t \tag{35}$$

where $\dot{Q}^{-1}$ is the inverse of $\dot{Q}$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ are constants.

Proof Let us use the notation from Chapter 2. From the assumptions regarding the functions Q and D it follows that $f, g \in C^1\left([0,T] \times \mathbb{R}^2\right)$, $\dot{Q}$ is strictly increasing on $\mathbb{R}$ and $\tilde{F}$ is concave on $C^1[0,T]$. The rest of the proof is analogous to proofs of Propositions 9 and 10. By Theorem 8, any function y that belongs to the feasible set and solves the Euler–Lagrange equation is a solution. From the assumptions it follows that the integral in (35) exists for any constants γ_1, γ_2. But without any knowledge about the functions D, Q we cannot determine if the boundary condition and the isoperimetric constraint can be satisfied.

Remark For $D(t) = \mathrm{e}^{-\rho t}$, $\rho \in \mathbb{R}_0^+$, and $Q(x) = x^2$ we obtain the problems from Sections 3.1 and 3.2, respectively.

3.3 Introducing additional constrains

In Section 3.2 we outlined the problem of minimization of the discounted resource variation cost. In line with § 6, 7 and 8 of Cullingford and Prideaux (1973) we introduce additional constraints to the problem, pertaining to the amount of work that must be done at some point of time.

3.3.1 A single equality constraint

Definition Let J be an interval. Denote f_J the restriction of function f to interval J.

Definition Let $T \in (0,1), \rho \in (0,\infty), W \in \mathbb{R}$. Define the set S as

$$S = \left\{ y \,\middle|\, y : [0,1] \to \mathbb{R}, y_{[0,T]} \in C^1[0,T], y_{[T,1]} \in C^1[T,1] \right\}. \qquad (36)$$

Note that $C^1[0,1] \subset S \subset C[0,1]$.

We consider the following problem. Find a maximizer of the functional $F : S \to \mathbb{R}$, defined as

$$F(y) = \int_0^T -\mathrm{e}^{-\rho t} \dot{y}_{[0,T]}^2(t)\ \mathrm{d}t + \int_T^1 -\mathrm{e}^{-\rho t} \dot{y}_{[T,1]}^2(t)\ \mathrm{d}t, \tag{37}$$

subject to the following constraints

$$y(0) = 0 \tag{38}$$

$$y(1) = 0 \tag{39}$$

$$\int_0^T y(t)\ \mathrm{d}t = W \tag{40}$$

$$\int_0^1 y(t)\ \mathrm{d}t = 1. \tag{41}$$

The values of T and W characterize an additional constraint imposed on the optimal resource allocation in the sense that exactly the share W of work must be done by time T.

One might also think of a resource allocation problem where *at least* W of work must be done by time T (or, for instance, where *no more than* a particular share of resources has been used up by time T). In such situation the constraint may become "inactive" in the sense that its removal it will not affect the solution. To determine whether the constraint is in fact inactive we must solve for both the constrained problem (with the corresponding *equality* constraint, as stated above) and the unconstrained problem (Section 3.2) and compare the respective values of the objective function.

Inactive constraints are dealt with in detail in Section 3.3.3.

Proposition 14 There exists exactly one function y from S, the formula of which is given below[5], that maximizes functional (37) subject to constraints (38) to (41).

Proof Let $x \in \mathbb{R}$. We find functions $y_{[0,T]}^x \in C^1[0,T]$ and $y_{[T,1]}^x \in C^1[T,1]$ that are maximizers of the respective integrals in (37) subject to constraints (38) to (41) *and*

[5] The formula is given in (42) with the parameters given in (45), (46) and (56).

the constraint $y^x_{[0,T]}(T) = y^x_{[T,1]}(T) = x$. This is the problem of minimization of the discounted resource variation cost as outlined in Section 3.2. By Proposition 10 it has exactly one solution. The resulting function $y^x \in S$, unique for each $x \in \mathbb{R}$, is a maximizer of (37) subject to constraints (38) to (41) *and* the constraint $y^x(T) = x$. If there exists $\hat{x} \in \mathbb{R}$ for which the value of $F(y^x)$ is maximal then $y^{\hat{x}}$ is the solution to the problem.

The Euler–Lagrange equation pertaining to the discounted resource variation cost and its solution, outlined in equations (29) to (31), apply. For any $x \in \mathbb{R}$ there exist $A_x, B_x, C_x, P_x, Q_x, R_x \in \mathbb{R}$ such that

$$y^x(t) = \begin{cases} A_x t \mathrm{e}^{\rho t} + B_x \mathrm{e}^{\rho t} + C_x & \text{for} \quad t \in [0, T] \\ P_x t \mathrm{e}^{\rho t} + Q_x \mathrm{e}^{\rho t} + R_x & \text{for} \quad t \in [T, 1]. \end{cases} \tag{42}$$

The constants $A_x, \ldots, R_x$ must be such that constraints (38) to (41) and the constraint $y^x(T) = x$ be satisfied, that is

$$\begin{bmatrix} y^x(0) \\ y^x(T) \\ \int_0^T y^x(t)\,\mathrm{d}t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ T\mathrm{e}^{\rho T} & \mathrm{e}^{\rho T} & 1 \\ \frac{\mathrm{e}^{\rho T}(\rho T - 1) + 1}{\rho^2} & \frac{\mathrm{e}^{\rho T} - 1}{\rho} & T \end{bmatrix} \begin{bmatrix} A_x \\ B_x \\ C_x \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ W \end{bmatrix} \tag{43}$$

$$\begin{bmatrix} y^x(T) \\ y^x(1) \\ \int_T^1 y^x(t)\,\mathrm{d}t \end{bmatrix} = \begin{bmatrix} T\mathrm{e}^{\rho T} & \mathrm{e}^{\rho T} & 1 \\ \mathrm{e}^{\rho} & \mathrm{e}^{\rho} & 1 \\ \frac{(\rho - 1)\mathrm{e}^{\rho} - (\rho T - 1)\mathrm{e}^{\rho T}}{\rho^2} & \frac{\mathrm{e}^{\rho} - \mathrm{e}^{\rho T}}{\rho} & T \end{bmatrix} \begin{bmatrix} P_x \\ Q_x \\ R_x \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 1 - W \end{bmatrix}. \tag{44}$$

The matrices in (43) and (44) are non-singular. After some tedious calculations we obtain their inverses, yielding, for a given x, the unique solution

$$\begin{bmatrix} A_x \\ B_x \\ C_x \end{bmatrix} = \begin{bmatrix} \frac{\rho\left(e^{\rho T}-\rho T-1\right)}{(e^{\rho T}-1)^2-\rho^2T^2e^{\rho T}} & \frac{\rho^2\left(1-e^{\rho T}\right)}{(e^{\rho T}-1)^2-\rho^2T^2e^{\rho T}} \\ -\frac{e^{\rho T}(\rho T-1)+1}{(e^{\rho T}-1)^2-\rho^2T^2e^{\rho T}} & \frac{\rho^2 T e^{\rho T}}{(e^{\rho T}-1)^2-\rho^2T^2e^{\rho T}} \\ \frac{e^{\rho T}(\rho T-1)+1}{(e^{\rho T}-1)^2-\rho^2T^2e^{\rho T}} & -\frac{\rho^2 T e^{\rho T}}{(e^{\rho T}-1)^2-\rho^2T^2e^{\rho T}} \end{bmatrix} \begin{bmatrix} x \\ W \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} P_x \\ Q_x \\ R_x \end{bmatrix} = \begin{bmatrix} \frac{\rho^2e^{\rho}(1-T)-\rho\left(e^{\rho}-e^{\rho T}\right)}{(e^{\rho}-e^{\rho T})^2-\rho^2(1-T)^2e^{\rho(T+1)}} & \frac{-\rho^2\left(e^{\rho}-e^{\rho T}\right)}{(e^{\rho}-e^{\rho T})^2-\rho^2(1-T)^2e^{\rho(T+1)}} \\ \frac{(\rho-1)e^{\rho}-(\rho T-1)e^{\rho T}-\rho^2(1-T)e^{\rho}}{(e^{\rho}-e^{\rho T})^2-\rho^2(1-T)^2e^{\rho(T+1)}} & \frac{\rho^2\left(e^{\rho}-Te^{\rho T}\right)}{(e^{\rho}-e^{\rho T})^2-\rho^2(1-T)^2e^{\rho(T+1)}} \\ \frac{e^{\rho}\left(e^{\rho}+e^{\rho T}(\rho T-\rho-1)\right)}{(e^{\rho}-e^{\rho T})^2-\rho^2(1-T)^2e^{\rho(T+1)}} & \frac{\rho^2e^{\rho(T+1)}(T-1)}{(e^{\rho}-e^{\rho T})^2-\rho^2(1-T)^2e^{\rho(T+1)}} \end{bmatrix} \begin{bmatrix} x \\ 1-W \end{bmatrix}. \tag{46}$$

Let us evaluate $F\left(y^x\right)$. From (37) and (42) we get

$$\begin{aligned} &\int_0^T -e^{-\rho t}\left(\dot{y}^x_{[0,T]}(t)\right)^2 \mathrm{d}t = \\ &= -\int_0^T \begin{bmatrix} \rho^2A_x^2 \\ 2\rho A_x\left(A_x+\rho B_x\right) \\ \left(A_x+\rho B_x\right)^2 \end{bmatrix}^T \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} e^{\rho t}\,\mathrm{d}t = \\ &= -\frac{1}{\rho^3}\begin{bmatrix} \rho^2A_x^2 \\ 2\rho A_x\left(A_x+\rho B_x\right) \\ \left(A_x+\rho B_x\right)^2 \end{bmatrix}^T \begin{bmatrix} \rho^2 & -2\rho & 2 \\ 0 & \rho^2 & -\rho \\ 0 & 0 & \rho^2 \end{bmatrix} \begin{bmatrix} T^2e^{\rho T} \\ Te^{\rho T} \\ e^{\rho T}-1 \end{bmatrix} = \\ &= -\begin{bmatrix} \frac{\rho^2T^2e^{\rho T}+e^{\rho T}-1}{\rho} \\ \rho T e^{\rho T} \\ \rho\left(e^{\rho T}-1\right) \end{bmatrix}^T \begin{bmatrix} A_x^2 \\ 2A_xB_x \\ B_x^2 \end{bmatrix} \end{aligned} \tag{47}$$

and it is similarly shown that

$$\int_T^1 -e^{-\rho t}\left(\dot{y}^x_{[T,1]}(t)\right)^2 \mathrm{d}t = -\begin{bmatrix} \frac{\left(\rho^2+1\right)e^{\rho}-\left(\rho^2T^2+1\right)e^{\rho T}}{\rho} \\ \rho\left(e^{\rho}-Te^{\rho T}\right) \\ \rho\left(e^{\rho}-e^{\rho T}\right) \end{bmatrix}^T \begin{bmatrix} P_x^2 \\ 2P_xQ_x \\ Q_x^2 \end{bmatrix}. \tag{48}$$

From (37), (47) and (48) it immediately follows that $F\left(y^x\right)$ is a sum of two binary quadratic forms in A_x, B_x and P_x, Q_x, respectively, and from (45) and (46) it follows that A_x, B_x, P_x, Q_x are linear functions of x. Therefore the function $x \to F(y^x)$ is quadratic. It can be shown that $x \to \infty$ or $-\infty$ would cause $F(y^x)$ to approach minus infinity, therefore the function must be strictly concave on $\mathbb{R}$ and must have exactly one point of maximum, $\hat{x}$, on $\mathbb{R}$, QED.

Remark The formula for $\hat{x}$ is derived in the Appendix (pages 37–39).

Figures (B) and (C) on pages 24 and 25, respectively, show graphs of maximizers y for different values of W and T.

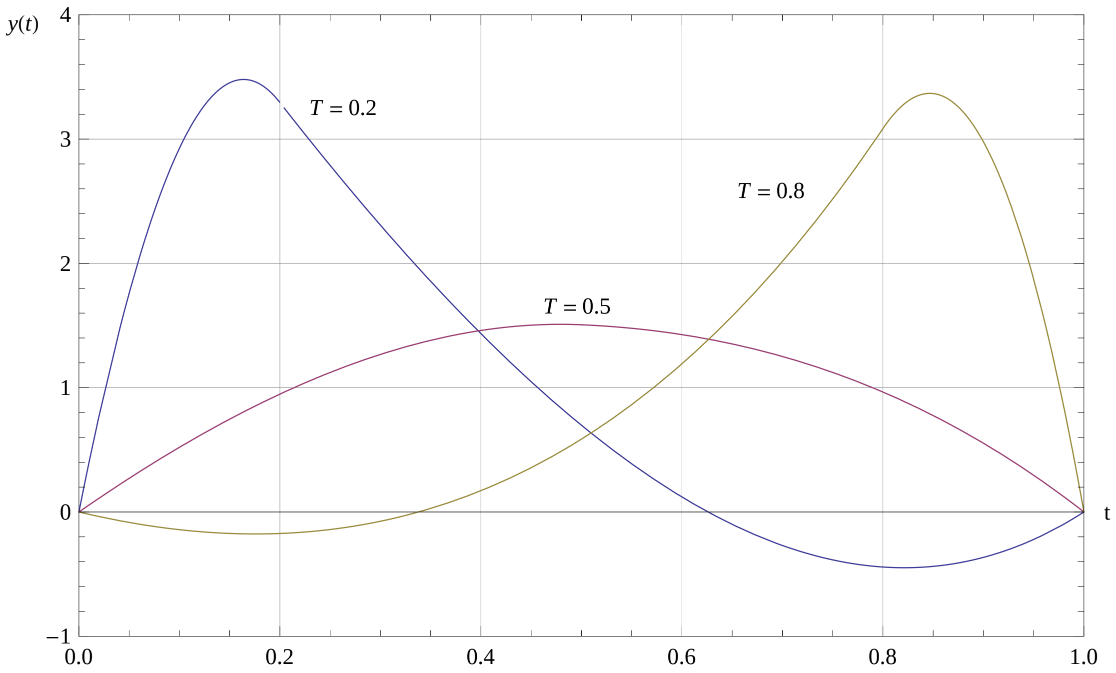

Figure (B): Optimal resource profiles y for $\rho = 0.7$, $W = 0.5$ and different values of T. They represent the optimal resource allocation under the requirement that half of work must be done by $\frac{1}{5}$ (or $\frac{1}{2}$, $\frac{4}{5}$, respectively) of the total time

3.3.2 Multiple equality constraints

The "constrained" problem of minimization of the resource variation cost, as stated in Section 3.3.1, can of course be extended to multiple constraints. In general the problem can be stated as follows.

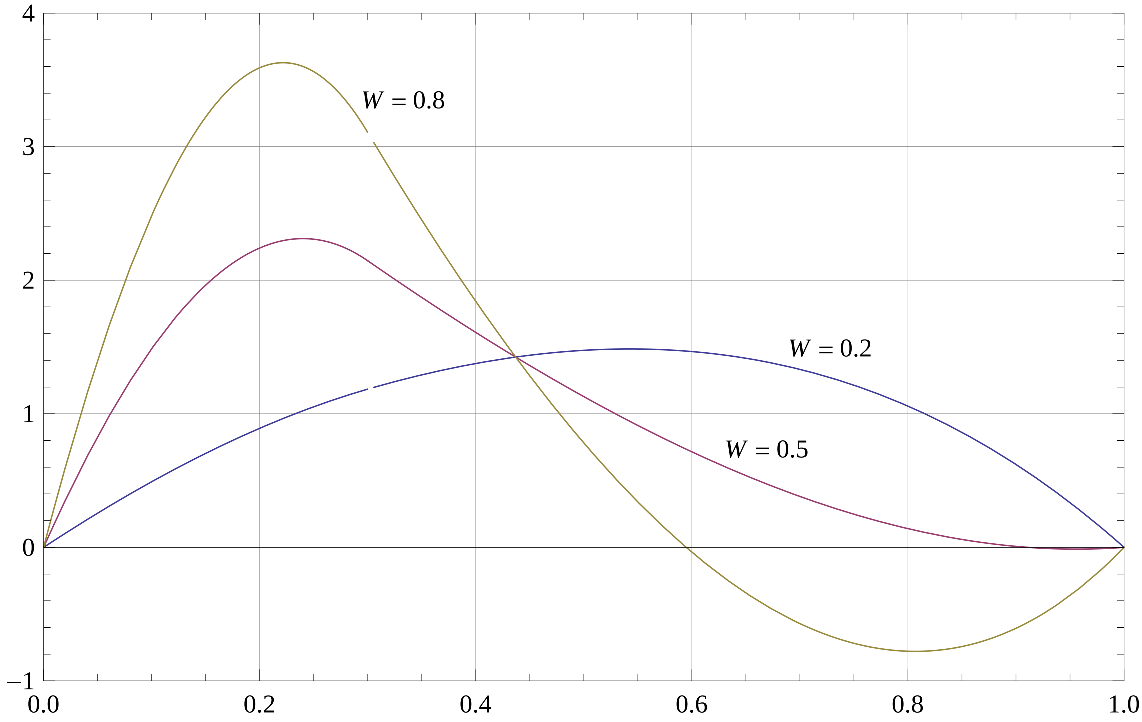

Figure (C): Optimal resource profiles y for $\rho = 0.7,\ T = 0.3$ and different values of W. They represent the optimal resource allocation under the requirement that exactly $\frac{1}{5}$ $\left(\text{or } \frac{1}{2}, \frac{4}{5},\right.$ respectively$\left.\right)$ of the total work must be done by time 0.3.

Let $n \in \mathbb{N}, \rho \in (0, \infty), A, B \in \mathbb{R}$, let $\mathbf{T}, \mathbf{W} \in \mathbb{R}^n$ be such that $0 < T_1 < T_2 < \ldots < T_n$ and define $T_0 = 0$. Let us define the set $\hat{S}$ as

$$\hat{S} = \left\{ y \,\middle|\, y : [0, T_n] \to \mathbb{R}, \forall i \in \{1, \ldots, n\} : y_{[T_{i-1}, T_i]} \in C^1 [T_{i-1}, T_i] \right\}.$$

Find a maximizer y_0 of the functional $F : \hat{S} \to \mathbb{R}$, defined as

$$F(y) = \sum_{i=1}^{n} \int_{T_{i-1}}^{T_i} -\mathrm{e}^{-\rho t} \dot{y}^2_{[T_{i-1}, T_i]}(t)\ \mathrm{d}t, \tag{49}$$

subject to

$$y(0) = A \tag{50}$$

$$y(T_n) = B \tag{51}$$

$$\forall i \in \{1, \ldots, n\} \quad \int_0^{T_i} y(t)\ \mathrm{d}t = W_i. \tag{52}$$

Again, the values of T_i and W_i determine that under the optimal work profile y_0 exactly W_i of work must be done by time T_i. Note that W_i need not be in any particular order ("negative work" is allowed). In the last subsection we explain how the introduction of inequality constraints, which may also appear among the class of optimization problems, complicates the solution.

3.3.3 Inequality constraints

If, for some i, we substitute "=" in (52) by "$\geq$" or "$\leq$", some of these constraints might become inactive.

The process of detecting inactive constraints is straightforward, yet merits explanation. All other things held constant, the value of $F(y_0)$ is a continuous function of the vector $\mathbf{W}$. If the constraints constitute a closed set in $\mathbb{R}^n$ where n is the number of constraints (e.g., they are in the form of *not* strict inequalities, closed intervals, or equalities) then a maximum of the objective function must exist on the feasible set.

An explicit formula for y_0 in terms of the given equality constraints can be found[6]—it is obvious that it is only the parameters and the equality constraints that affect the solution. If a problem involving an inequality constraint is given and a solution exists then it satisfies the constraint either as an equality[7] or as a strict inequality. But it has been said that equality constraints are the only ones upon which the formula for y_0 depends. Thus in the latter case the constraint is inactive

[6] Its derivation does not differ much from the single-constraint case.

[7] Given an appropriate metric space that would contain the feasible set, we may say that the solution lies on the boundary of the set.

and can be removed from the problem without affecting the solution. The process then starts anew until all the remaining constraints are active. These are satisfied as equalities, thus can be treated as equality constraints.

It is not known *ex ante* which constraints are active but there are 2^n possible combinations, each yielding a candidate solution. We discard all those which do not belong to the feasible set. Finally, we have asserted that under certain assumptions a unique solution to the cost minimization problem indeed exists—obviously it is the one with the greatest value of the objective function.

4 Conclusion

In my bachelor's thesis, which deals with a problem in calculus of variations called the isoperimetric problem, I have presented several problems that seek to minimize cost associated with a particular resource allocation. The problems discussed in the thesis are of course only of limited applicability. There seem to be several main limitations.

First, the requirement that there be an explicit formula which assigns cost to each resource profile[8] is particularly stringent in cases where the link between production and costs is only empirical.

Second, the absence of uncertainty in the basic formulation of the isoperimetric problem effectively hinders any serious attempt to properly plan the future allocation of resources. This could be partly remedied by having the discount factor follow a stochastic process (e.g., to represent the fluctuation of interest rates) instead of being deterministic, let alone constant over time. The same could be done with the cost function itself (e.g., to reflect the uncertainty about the future cost of labour). The isoperimetric constraint could also include a stochastic variable[9], thus reflecting the variability in productivity of the resource in question.

Third, the formulation fails to account for discrete-choice applications[10].

The isoperimetric problem is nevertheless a useful method of optimization. The approaches mentioned in this thesis can serve as a basis for more elaborate models which would eventually be used in real-world situations requiring cost minimization.

(Following is a Czech translation of the above conclusion).

[8] in our case, $\int_0^T \mathrm{e}^{-\rho t} \dot{y}^2(t)\,\mathrm{d}t$

[9] e.g., $\int_0^T p(t)\,y(t)\,\mathrm{d}t = W$ where p would follow a stochastic process

[10] e.g., if $y(t)$ had to be chosen from a finite set $\{a_1, a_2, \dots, a_n\} \subset \mathbb{R}$ for each $t \in [0, T]$

Závěr

V bakalářské práci jsem se zabýval úlohou z variačního počtu zvanou isoperimetrická úloha a představil několik problémů minimalizace nákladů řízení zdrojů. Možnost jejich aplikace je nicnémě omezená z několika důvodů.

Zaprvé, v případech, kdy vazba mezi výrobou a náklady je pouze empirická, často nelze nákladovou funkci exaktně vyjádřit, což koliduje s předpoklady úlohy.

Zadruhé, jelikož základní formulace úlohy, představená v práci, pracuje pouze s deterministickými veličinami, nelze úlohu spolehlivě aplikovat za podmínek nejistoty. Tento nedostatek lze částečně eliminovat nahrazením konstantního či jakkoli jinak deterministického diskontního faktoru stochastickou veličinou (což může reprezentovat pohyb úrokových sazeb), analogickou úpravou nákladové funkce (což by odráželo nejistotu ohledně budoucích cen práce) či úpravou isoperimetrické podmínky tak, aby odrážela např. změny produktivity zdroje.

Zatřetí, požadavek na spojité řešení eliminuje případy, kdy je alokace zdrojů možná pouze jako diskrétní veličina.

Isoperimetrickou úlohu lze přesto považovat za užitečnou optimalizační metodu. Přístupy zmíněné v této práci mohou sloužit jako základ případných složitějších modelů majících za cíl minimalizaci nákladů v konkrétních situacích.

5 References

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6 Appendix

6.1 Linear approximations

Let $t \in [0, T]$. We are interested in the first-order Taylor expansion of the function $y : [0, T] \times [0, \infty) \to \mathbb{R}$, defined as

$$y(t, \rho) = y_\rho(t) = \begin{cases} \frac{6W}{T^3}(T - t)t & \text{for} \quad \rho = 0 \\ W\rho^2 \frac{\left(e^{\rho t}-1\right)Te^{\rho T} - \left(e^{\rho T}-1\right)te^{\rho t}}{\left(e^{\rho T}-1\right)^2 - \rho^2 T^2 e^{\rho T}} & \text{for} \quad \rho > 0, \end{cases} \tag{53}$$

about the point $[t, 0_+]$.

Proposition 15 For all $t \in [0, T]$,

$$\lim_{\rho \to 0+} y(t, \rho) = y(t, 0).$$

Proof The computation of the limit is rather straightforward but requires much space due the complicated nature of the function in question. Define $a_t, f_t, g : \mathbb{R} \to \mathbb{R}$ as follows

$$\begin{aligned} a_t(\rho) &= \left(e^{\rho t} - 1\right)Te^{\rho T} - \left(e^{\rho T} - 1\right)te^{\rho t}, \\ f_t(\rho) &= W\rho^2 a_t(\rho), \\ g(\rho) &= \left(e^{\rho T} - 1\right)^2 - \rho^2 T^2 e^{\rho T}. \end{aligned}$$

Then[11], for $\rho > 0$, $y(t, \rho) = \frac{f_t}{g}$. Let $l \in \mathbb{N}$ be the smallest number for which $g^{(l)}(0) \neq 0$. We use l times l'Hôpital's rule together with the fact that $a_t, f_t, g \in C^\infty(\mathbb{R})$ to write

$$\lim_{\rho \to 0+} y(t, \rho) = \lim_{\rho \to 0+} \frac{f_t(\rho)}{g(\rho)} = \lim_{\rho \to 0+} \frac{f_t^{(l)}(\rho)}{g^{(l)}(\rho)} = \frac{f_t^{(l)}(0)}{g^{(l)}(0)}. \tag{54}$$

[11]For the sake of clarity we shall occasionally write f_t instead of $f_t(\rho)$.

Let $n \in \mathbb{N} \cup \{0\}$. Let us compute the n^{th} derivatives[12] of a_t, f_t, g. The following formulae are easily proven by mathematical induction:

$$a_t^{(n)}(\rho) = (T-t)(T+t)^n \mathrm{e}^{\rho(T+t)} + t^{n+1}\mathrm{e}^{\rho t} - T^{n+1}\mathrm{e}^{\rho T},$$

$$f_t^{(n)}(\rho) = \begin{cases} W\rho^2 a_t(\rho) & \text{for } n = 0, \\ W\left[\rho^2 a_t'(\rho) + 2\rho a_t(\rho)\right] & \text{for } n = 1, \\ W\left[\rho^2 a_t^{(n)}(\rho) + 2n\rho a_t^{(n-1)}(\rho) + n(n-1)\, a_t^{(n-2)}(\rho)\right] & \text{for } n \geq 2, \end{cases}$$

$$g^{(n)}(\rho) = T^n \mathrm{e}^{\rho T}\left(2^n \mathrm{e}^{\rho T} - \rho^2 T^2 - 2n\rho T - n(n-1) - 2\right) + j,$$

where $j = 1$ if $n = 0$, otherwise $j = 0$. For $\rho = 0$ we obtain

$$a_t^{(n)}(0) = \sum_{i=1}^{n} \frac{n!\,(n-2i+1)!}{i!\,(n-i+1)!} T^{n-i+1} t^i,$$

$$f_t^{(n)}(0) = \begin{cases} 0 & \text{for } n \in \{0, 1\}, \\ \sum_{i=1}^{n-2} \frac{n!(n-2i-1)!}{i!(n-i-1)!} W T^{n-i-1} t^i & \text{for } n \geq 2, \end{cases}$$

$$g^{(n)}(0) = \left(2^n - 2 - n(n-1)\right) T^n + j,$$

which is, for a few particular values of n, equal to

n	$f_t^{(n)}(0)$	$g^{(n)}(0)$
0	0	0
1	0	0
2	0	0
3	0	0
4	$12WTt\,(T-t)$	$2T^4$
5	$40WTt\,(T^2-t^2)$	$10T^5$.

[12] $f_t^{(n)}$ will denote the n^{th} derivative of f_t with $f_t^{(0)}$ being the same as f_t for convenience.

From this and from (54) we obtain

$$\lim_{\rho\to 0+} y\left(t,\rho\right) = \frac{f_t^{(4)}\left(0\right)}{g^{(4)}\left(0\right)} = \frac{6W}{T^3}\left(T-t\right)t,$$

QED.

Lemma 16 Let $k:\mathbb{R}\to\mathbb{R}$ be right-continuous at point $a\in\mathbb{R}$. If $\lim_{x\to a+} k'\left(x\right)$ exists then also $k'\left(a_+\right)$ exists and the two are equal.

Proposition 17 For all $t\in[0,T]$,

$$\partial_2 y\left(t,0_+\right) = 4W\frac{t}{T}\left(1-\frac{t}{T}\right)\left(\frac{t}{T}-\frac{1}{2}\right).$$

Proof Let $N_t, D:\mathbb{R}\to\mathbb{R}$, $N_t\left(\rho\right) = f_t'\left(\rho\right)g\left(\rho\right) - f_t\left(\rho\right)g'\left(\rho\right)$, $D\left(\rho\right) = g^2\left(\rho\right)$. Then $\partial_2 y\left(t,\rho\right) = \frac{N_t(\rho)}{D(\rho)}$. Obviously $N_t, D\in C^\infty\left(\mathbb{R}\right)$. From Proposition 15 it follows that y is continuous on $[0,T]\times[0,\infty)$, thus right-continuous in ρ at $\rho=0$. Let $m\in\mathbb{N}$ be the smallest number for which $D^{(m)}\left(0\right)\neq 0$. We use Lemma 16 and m times l'Hôpital's rule to obtain

$$\partial_2 y\left(t,0_+\right) = \lim_{\rho\to 0+}\partial_2 y\left(t,\rho\right) = \lim_{\rho\to 0+}\frac{N_t\left(\rho\right)}{D\left(\rho\right)} = \lim_{\rho\to 0+}\frac{N_t^{(m)}\left(\rho\right)}{D^{(m)}\left(\rho\right)} = \frac{N_t^{(m)}\left(0\right)}{D^{(m)}\left(0\right)}. \tag{55}$$

Let a_t, f_t, g be the same as before. Let us compute the n^{th} derivatives of N_t, D:

n	$N_t^{(n)}(\rho)$	$N_t^{(n)}(0)$
0	$f_t'g - f_tg'$	0
1	$f_t''g - f_tg''$	0
2	$f_t^{(3)}g + f_t''g' - f_t'g'' - f_tg^{(3)}$	0
3	$f_t^{(4)}g + 2f_t^{(3)}g - 2f_t'g^{(3)} - f_tg^{(4)}$	0
4	$f_t^{(5)}g + 3f_t^{(4)}g' + 2f_t^{(3)}g'' - 2f_t''g^{(3)} - 3f_t'g^{(4)} - f_tg^{(5)}$	0
$\vdots$	$\vdots$	$\vdots$
8	$f_t^{(9)}g + 7f_t^{(8)}g' + 20f_t^{(7)}g'' + 28f_t^{(6)}g^{(3)} + 14f_t^{(5)}g^{(4)} - 14f_t^{(4)}g^{(5)} - 28f_t^{(3)}g^{(6)} - 20f_t''g^{(7)} - 7f_t'g^{(8)} - f_tg^{(9)}$	$560WT^5t\left(-T^2 + +3Tt - 2t^2\right)$

and

n	$D^{(n)}(\rho)$	$D^{(n)}(0)$
0	g^2	0
1	$2gg'$	0
2	$2(g')^2 + 2gg''$	0
3	$6g'g'' + 2gg^{(3)}$	0
4	$6(g'')^2 + 8g'g^{(3)} + 2gg^{(4)}$	0
5	$20g''g^{(3)} + 10g'g^{(4)} + 2gg^{(5)}$	0
6	$20\left(g^{(3)}\right)^2 + 30g''g^{(4)} + 12g'g^{(5)} + 2gg^{(6)}$	0
7	$70g^{(3)}g^{(4)} + 42g''g^{(5)} + 14g'g^{(6)} + 2gg^{(7)}$	0
8	$70\left(g^{(4)}\right)^2 + 112g^{(3)}g^{(5)} + 56g''g^{(6)} + 16g'g^{(7)} + 2gg^{(8)}$	$280T^8$.

From this and from (55) we obtain

$$\partial_2 y\left(t, 0_+\right) = \frac{N_t^{(8)}(0)}{D^{(8)}(0)} = 2\frac{W}{T^3}t\left(-T^2 + 3Tt - 2t^2\right)$$

from which quickly follows the form that appears in Proposition 17, QED.

Proof of Proposition 11 (page 19) follows directly from Propositions 15 and 17 and from the properties of the Taylor expansion.

Definition Define $\Upsilon : [0, \infty) \to \mathbb{R}$ as

$$\Upsilon(\rho) = -F_\rho(y_\rho) = \int_0^T \mathrm{e}^{-\rho t} (\partial_1 y(t, \rho))^2 \, \mathrm{d}t = \begin{cases} 12\frac{W^2}{T^3} & \text{for } \rho = 0, \\ \frac{W^2\rho^3\left(\mathrm{e}^{\rho T}-1\right)}{\left(\mathrm{e}^{\rho T}-1\right)^2-\rho^2T^2\mathrm{e}^{\rho T}} & \text{for } \rho > 0. \end{cases}$$

Proposition 18

$$\lim_{\rho\to 0+} \Upsilon(\rho) = \Upsilon(0).$$

Proof Let g be the same as in the proof of Proposition 15. Let $h(\rho) = W^2\rho^3\left(\mathrm{e}^{\rho T} - 1\right)$. Then, for $\rho > 0$, $\Upsilon(\rho) = \frac{h}{g}$ and the rest is analogous to the proof of Proposition 15. For $n \in \mathbb{N} \cup \{0\}$

$$h^{(n)}(\rho) = W^2T^{n-3}\mathrm{e}^{\rho T}\left(\rho^3T^3 + 3n\rho^2T^2 + 3n(n-1)\rho T + n(n-1)(n-2)\right),$$

$$h^{(n)}(0) = n(n-1)(n-2)W^2T^{n-3} + \begin{cases} -6W^2 & \text{if } n = 3, \\ 0 & \text{if } n \geq 4. \end{cases}$$

For the first few n we have

n	$h^{(n)}(0)$	$g^{(n)}(0)$
0	0	0
1	0	0
2	0	0
3	0	0
4	$24W^2T$	$2T^4$
5	$60W^3T^2$	$10T^5$

and finally

$$\lim_{\rho\to 0+} \Upsilon(\rho) = \frac{h^{(4)}(0)}{g^{(4)}(0)} = 12\frac{W^2}{T^3},$$

QED.

Proposition 19

$$\Upsilon'(0_+) = -6\frac{W^2}{T^2}.$$

Proof According to Proposition 18, Υ is right-continuous at 0. Due to Lemma 16 it suffices to calculate the limit

$$\Upsilon'(0_+) = \lim_{\rho\to 0+} \Upsilon'(\rho) = \lim_{\rho\to 0+} \frac{h'(\rho)\, g(\rho) - h(\rho)\, g'(\rho)}{g^2(\rho)}.$$

The rest is analogous to the proof of Proposition 17 with h assuming the place of f_t. Eventually we obtain

$$\Upsilon'(0_+) = \frac{14\left(h^{(5)}(0)\, g^{(4)}(0) - h^{(4)}(0)\, g^{(5)}(0)\right)}{70\left(g^{(4)}(0)\right)^2} = -6\frac{W^2}{T^2},$$

QED.

Proof of Proposition 12 (page 19) follows directly from Propositions 18 and 19 and from the properties of the Taylor expansion.

6.2 The optimal value of $\hat{x} = y(T)$

Proposition 20 Let $\hat{x}$ be defined as in Section 3.3.1. Then

$$\hat{x} = -\frac{\begin{array}{c}(v_1 m_1 m_2 + v_2 m_1 m_4 + v_2 m_3 m_2 + v_3 m_3 m_4)\, W + \\ + (v_4 m_5 m_6 + v_5 m_5 m_8 + v_5 m_7 m_6 + v_6 m_7 m_8)(1 - W)\end{array}}{v_1 m_1^2 + 2 v_2 m_1 m_3 + v_3 m_3^2 + v_4 m_5^2 + 2 v_5 m_5 m_7 + v_6 m_7^2}. \tag{56}$$

where

$$m_1 = \frac{\rho\left(e^{\rho T}-\rho T-1\right)}{\left(e^{\rho T}-1\right)^2-\rho^2T^2e^{\rho T}} \qquad m_2 = \frac{\rho^2\left(1-e^{\rho T}\right)}{\left(e^{\rho T}-1\right)^2-\rho^2T^2e^{\rho T}}$$

$$m_3 = \frac{e^{\rho T}(\rho T-1)+1}{\left(e^{\rho T}-1\right)^2-\rho^2T^2e^{\rho T}} \qquad m_4 = -\frac{\rho^2Te^{\rho T}}{\left(e^{\rho T}-1\right)^2-\rho^2T^2e^{\rho T}}$$

$$m_5 = \frac{\rho^2e^{\rho}(1-T)-\rho\left(e^{\rho}-e^{\rho T}\right)}{\left(e^{\rho}-e^{\rho T}\right)^2-\rho^2(1-T)^2e^{\rho(T+1)}} \qquad m_6 = \frac{-\rho^2\left(e^{\rho}-e^{\rho T}\right)}{\left(e^{\rho}-e^{\rho T}\right)^2-\rho^2(1-T)^2e^{\rho(T+1)}}$$

$$m_7 = \frac{(\rho-1)e^{\rho}-(\rho T-1)e^{\rho T}-\rho^2(1-T)e^{\rho}}{\left(e^{\rho}-e^{\rho T}\right)^2-\rho^2(1-T)^2e^{\rho(T+1)}} \qquad m_8 = \frac{\rho^2\left(e^{\rho}-Te^{\rho T}\right)}{\left(e^{\rho}-e^{\rho T}\right)^2-\rho^2(1-T)^2e^{\rho(T+1)}}$$

and

$$v_1 = \frac{\rho^2T^2e^{\rho T}+e^{\rho T}-1}{\rho} \qquad v_4 = \frac{\left(\rho^2+1\right)e^{\rho}-\left(\rho^2T^2+1\right)e^{\rho T}}{\rho}$$

$$v_2 = \rho Te^{\rho T} \qquad v_5 = \rho\left(e^{\rho}-Te^{\rho T}\right)$$

$$v_3 = \rho\left(e^{\rho T}-1\right) \qquad v_6 = \rho\left(e^{\rho}-e^{\rho T}\right).$$

Proof From the properties of the function $x \to F(y^x)$ it follows that it has zero derivative at the point $\hat{x}$. From equations (45) to (48) we obtain

$$-F(y^x) = v_1A_x^2 + 2v_2A_xB_x + v_3B_x^2 + v_4P_x^2 + 2v_5P_xQ_x + v_6Q_x^2.$$

Note that m_i are the elements of the matrices in (45) and (46) and v_i are elements of the vectors in (47) and (48). Obviously

$$A_x = m_1x + m_2W \qquad A'_x = m_1$$

$$B_x = m_3x + m_4W \qquad B'_x = m_3$$

$$P_x = m_5x + m_6(1-W) \qquad P'_x = m_5$$

$$Q_x = m_7x + m_8(1-W) \qquad Q'_x = m_7,$$

therefore

$$-\tfrac{1}{2}\frac{\partial}{\partial x}F\left(y^x\right)=v_1A_xA'_x+v_2\left(A'_xB_x+A_xB'_x\right)+v_3B_xB'_x+$$
$$+v_4P_xP'_x+v_5\left(P'_xQ_x+P_xQ'_x\right)+v_6Q_xQ'_x=$$
$$=\left(v_1m_1^2+2v_2m_1m_3+v_3m_3^2+v_4m_5^2+2v_5m_5m_7+v_6m_7^2\right)x+$$
$$+\left(v_1m_1m_2+v_2m_1m_4+v_2m_3m_2+v_3m_3m_4\right)W+$$
$$+\left(v_4m_5m_6+v_5m_5m_8+v_5m_7m_6+v_6m_7m_8\right)(1-W)$$

from which (56) immediately follows, QED.