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Fair Voting Rules in Committees, Strict Proportional Power and Optimal Quota

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Abstract In simple weighted committees with a finite number of members, fixed weights and changing quota, there exist a finite number of different quota intervals of stable power with the same sets of winning coalitions for all quotas from each of them. If in a committee the sets of winning coalitions for different quotas are the same, then the power indices based on pivots, swings, or minimal winning coalitions are also the same for those quotas. If the fair distribution of voting weights is defined, then the fair distribution of voting power means to find a quota that minimizes the distance between relative voting weights and relative voting power (optimal quota). The problem of the optimal quota has an exact solution via the finite number of quotas from different intervals of stable power.

JEL Classifications C71, D72, H77
Keywords simple weighted committee, fairness, optimal quota, strict proportional power, voting and power indices

1. Introduction

Let us consider a committee with n members. Each member has some voting weight (number of votes, shares etc.) and a voting rule is defined by a minimal number of weights required for passing a proposal. Given a voting rule, voting weights provide committee members with voting power. Voting power means an ability to influence the outcome of voting. Voting power indices are used to quantify the voting power.

The concept of fairness is being discussed related to the distribution of voting power among different actors of voting. This problem was clearly formulated by Nurmi (1982): "If one aims at designing collective decision-making bodies which are democratic in the sense of reflecting the popular support in terms of the voting power, we need indices of the latter which enable us to calculate for any given distribution of support and for any

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decision rule the distribution of seats that is 'just'. Alternatively, we may want to design decision rules that - given the distribution of seats and support - lead to a distribution of voting power which is identical with the distribution of support" (Nurmi 1982: 204).

Voting power is not directly observable: as a proxy for it voting weights are used. Therefore, fairness is usually defined in terms of voting weights (e.g. voting weights are proportional to the results of an election). Assuming that a principle of fair distribution of voting weights is selected, we are addressing the question of how to achieve equality of voting power (at least approximately) to voting weights. The concepts of strict proportional power and the randomized decision rule introduced by Holler (1982a, 1985, 1987), of optimal quota of Słomczyński and Życzkowski (2007), and of intervals of stable power (Turnovec 2008b) are used to find, given voting weights, a voting rule minimizing the distance between actors' voting weights and their voting power.

Concept of fairness is frequently associated with so-called square root rule, attributed to British statistician Lionel Penrose (1946). The square root rule is closely related to indirect voting power measured by the Penrose-Banzhaf power index.[1] Different aspects of the square root rule have been analysed in Felsenthal and Machover (1998, 2004), Laruelle and Widgrén (1998), Baldwin and Widgrén (2004), Turnovec (2009). The square root rule of "fairness" in the EU Council of Ministers voting was discussed and evaluated in Felsenthal and Machover (2007), Słomczyński and Życzkowski (2006, 2007), Hosli (2008), Leech and Aziz (2008), Turnovec (2008a) and others. Nurmi (1997a) used it to evaluate the representation of voters' groups in the European Parliament.

In the second section basic definitions are introduced and the applied power indices methodology is shortly resumed. The third section introduces the concept of quota intervals of stable power and optimal quota. While the framework of the analysis of fairness is usually restricted to the Penrose-Banzhaf concept of power, we are treating it in a more general setting and our

[1] The square root rule is based on the following propositions: Let us assume n units with population $p_1, p_2, \ldots, p_n$, and the system of representation by a super-unit committee with voting weights $w_1, w_2, \ldots, w_n$ of units' representations. It can be rigorously proved that for sufficiently large p_i the absolute Penrose-Banzhaf power of individual citizen of unit i in unit's referendum is proportional to the square root of p_i. If the relative Penrose-Banzhaf voting power of unit i representation is proportional to its voting weight, then indirect voting power of each individual citizen of unit i is proportional to the product of voting weight w_i and square root of population p_i. Based on the conjecture (not rigorously proved) that for n large enough the relative voting power is proportional to the voting weights, the square root rule concludes that the voting weights of units representations in the super-unit committee proportional to square roots of units' population lead to the same indirect voting power of each citizen independently of the population unit she is affiliated with.

results are relevant for any power index based on pivots or swings and for any concept of fairness.

2. Committees and voting power

A simple weighted committee is a pair $[N, \mathbf{w}]$, where N will be a finite set of n committee members $i=1,2,...,n$, and $\mathbf{w}=(w_1, w_2,...,w_n)$ will be a non-negative vector of the committee members' voting weights (e.g. votes or shares). By 2^N we denote the power set of N (set of all subsets of N). By voting coalition we mean an element $S \in 2^N$, the subset of committee members voting uniformly (YES or NO), and

$$w(S)=\sum_{i \in S} w_i$$

denotes the voting weight of coalition S. The voting rule is defined by quota q satisfying $0 < q \leq w(N)$, where q represents the minimal total weight necessary to approve the proposal. Triple $[N,q,\mathbf{w}]$ we call a *simple quota weighted committee*. The (voting) coalition S in committee $[N,q,\mathbf{w}]$ is called a winning one if $w(S) \geq q$ and a losing one in the opposite case. The winning coalition S is called critical if there exists at least one member $k \in S$ such that $w(S \backslash k) < q$ (we say that k is critical in S). The winning coalition S is called minimal if any of its members is critical in S.

A priori voting power analysis seeks an answer to the following question: Given a simple quota weighted committee $[N,q,\mathbf{w}]$, what is an influence of its members over the outcome of voting? The absolute voting power of a member i is defined as a probability $\prod_i [N,q,\mathbf{w}]$ that i will be decisive in the sense that such a situation appears in which she would be able to decide the outcome of voting by her vote (Nurmi 1997b and Turnovec 1997), and a relative voting power as

$$\pi_i[N,q,w] = \frac{\prod_i [N,q,w]}{\sum_{k \in N} \prod_k [N,q,w]}$$

Three basic concepts of decisiveness are used: swing position, pivotal position and membership in a minimal winning coalition (MWC position). The s*wing position* is an ability of an individual voter to change the outcome of voting by a unilateral switch from YES to NO (if member j is critical with respect to a coalition S, we say that he has a swing in S). The *pivotal position is* such a position of an individual voter in a permutation of voters expressing a ranking of attitudes of members to the voted issue (from the most preferable to the least preferable) and the corresponding order of forming of the winning coalition, in which her vote YES means a YES outcome of voting and her vote NO means a NO outcome of voting (we say that j is pivotal in the permutation considered). The MWC position is an ability of an individual voter to contribute to a minimal winning coalition (i.e. membership in the minimal winning coalition).

Let us denote by $W(N,q,\mathbf{w})$ the set of all winning coalitions and by $W_i(N,q,\mathbf{w})$ the set of all winning coalitions with i as a member, $C(N,q,\mathbf{w})$ as the set of all critical winning coalitions, and by $C_i(N,q,\mathbf{w})$ the set of all critical winning coalitions i has the swing in, by $P(N,q,\mathbf{w})$ the set of all permutations of N and $P_i(N,q,\mathbf{w})$, the set of all permutations i is pivotal in. Moreover, $M(N,q,\mathbf{w})$ represents the set of all minimal winning coalitions, and $M_i(N,q,\mathbf{w})$ is the set of all minimal winning coalitions with i. By $\text{card}(S)$ we denote the cardinality of S, therefore $\text{card}(\varnothing)=0$.

Assuming many voting acts and all coalitions equally likely, it makes sense to evaluate the a priori voting power of each member of the committee by the probability to have a swing, measured by the absolute Penrose-Banzhaf (PB) power index (Penrose 1946, Banzhaf 1965):

$$\Pi_i^{PB}(N,q,w)=\frac{\text{card}(C_i)}{2^{n\text{-}1}}$$

Here, $\text{card}(C_i)$ is the number of all winning coalitions the member i has the swing in and 2^{n-1} is the number of all possible coalitions with i. To compare the relative power of different committee members, the relative form of the PB power index is used:

$$\pi_i^{PB}(N,q,w)=\frac{\text{card}(C_i)}{\sum\limits_{k\in N}\text{card}(C_k)}$$

While the absolute PB is based on a well-established probability model (see e.g. Owen 1972), its normalization (relative PB index) destroys this probabilistic interpretation, the relative PB index simply answers the question of what is the voter i's share in all possible swings.

Assuming many voting acts and all possible preference orderings equally likely, it makes sense to evaluate an a priori voting power of each committee member by the probability of being in pivotal situation, measured by the Shapley-Shubik (SS) power index (Shapley and Shubik 1954):

$$\Pi_i^{SS}(N,q,w) = \frac{\text{card}(P_i)}{n!}$$

Here, $\text{card}(P_i)$ is the number of all permutations in which the committee member i is pivotal, and $n!$ is the number of all possible permutations of committee members. Since $\sum_{i \in N} \text{card}(P_i) = n!$ it holds that

$$\pi_i^{SS}(N,q,\mathbf{w}) = \frac{\text{card}(P_i)}{\sum_{k \in N} \text{card}(P_k)} = \frac{\text{card}(P_i)}{n!}$$

i.e. the absolute and relative form of the SS-power index is the same.[2]

Assuming many voting acts and all possible coalitions equally likely, it makes sense to evaluate the voting power of each committee member by the probability of membership in a minimal winning coalition, measured by the absolute Holler-Packel (HP) power index:

$$\Pi_i^{HP}(N,q,\mathbf{w}) = \frac{\text{card}(M_i)}{2^n}$$

Here, $\text{card}(M_i)$ is the number of all minimal winning coalitions with i, and 2^n is the number of all possible coalitions.[3] Originally the HP index was defined and is usually being presented in its relative form (Holler 1982b, Holler and Packel 1983)

[2] Supporters of the Penrose-Banzhaf power concept sometimes reject the Shapley-Shubik index as a measure of voting power. Their objections to the Shapley-Shubik power concept are based on the classification of power measures on so-called I-power (voter's potential influence over the outcome of voting) and P power (expected relative share in a fixed prize available to the winning group of committee members, based on cooperative game theory) introduced by Felsenthal et al. (1998). The Shapley-Shubik power index was declared to represent P-power and as such is unusable for measuring influence in voting. We tried to show in Turnovec (2007) and Turnovec et al. (2008) that objections against the Shapley-Shubik power index, based on its interpretation as a P-power concept, are not sufficiently justified. Both Shapley-Shubik and Penrose-Banzhaf measure could be successfully derived as cooperative game values, and at the same time both of them can be interpreted as probabilities of being in some decisive position (pivot, swing) without using cooperative game theory at all.

[3] The definition of an absolute HP power index is provided by the author (a similar definition of absolute PB power can be found in Brueckner (2001), the only difference is that we relate the number of MWC positions of member i to the total number of coalitions, not to the number of coalitions of which i is a member).

$$\prod_i^{HP}(N,q,\mathbf{w}) = \frac{\text{card}(M_i)}{\sum_{k \in N} \text{card}(M_k)}$$

The above definition of the absolute HP index allows a clear probabilistic interpretation. Multiplying and dividing it by the $\text{card}(M)$, we obtain

$$\frac{\text{card}(M_i)}{\text{card}(M)} \frac{\text{card}(M)}{2^n}$$

In this breakdown the first term gives the probability of being a member of a minimal winning coalition, provided the MWC is formed, and the second term the probability of forming a minimal winning coalition assuming that all voting coalitions are equally likely. The relative HP index has the same problem with a probabilistic interpretation as the relative PB index.[4]

In the literature there are still two other concepts of power indices: the Johnston (J) power index based on swings, and the Deegan-Packel (DP) power index, based on membership in minimal winning coalitions.

The Johnston power index (Johnston 1978) measures the power of a member of a committee as a normalized weighted average of the number of her swings, using as weights the reciprocals of the total number of swings in each critical winning coalition. (The swing members of the same winning coalition have the same power, which is equal to $1/[\#$ of swing members].)

The Deegan-Packel power index (Deegan and Packel 1978) measures the power of a member of a committee as a normalized weighted average of the number of minimal critical winning coalitions he is a member of, using as weights the reciprocals of the size of each MWC

It is difficult to provide some intuitively acceptable probabilistic interpretation for relative J and DP power indices. They provide a normative scheme of the division of rents in the committee rather than a measure of an a priori power (and in the sense of Felsenthal and Machover (1998) classification they can be considered as measures of P-power).

It can be easily seen that for any $\alpha > 0$ and any power index based on swings, pivots or MWC positions it holds that $\prod_i[N,\alpha q,\alpha\mathbf{w}] = \prod_i[N,q,\mathbf{w}]$. Therefore, without the loss of generality, we shall assume throughout the text that

[4] For a discussion about the possible probabilistic interpretation of the relative PB and HP, see Widgrén (2001).

$$\sum_{i\in N} w_i = 1 \quad \text{and} \quad 0 < q \le 1$$

using only relative voting weights and quotas in the analysis.

3. Quota interval of stable power, fairness and optimal quota

Let us formally define some concepts that we shall use in this paper.

Definition 1 A simple weighted committee $[N,\mathbf{w}]$ has a property of *strict proportional power* with respect to a power index $\boldsymbol{\pi}$, if there exists a voting rule q^* such that $\boldsymbol{\pi}[N,q^*,\mathbf{w}] = \mathbf{w}$, i.e. the relative voting power of committee members is equal to their relative voting weights.

In general, there is no reason to expect that such a voting rule exists. Holler and Berg (1986) extended the concept of a strict proportional power in the model with randomized voting rule.

Definition 2 Let $[N,\mathbf{w}]$ be a simple weighted committee, $\mathbf{q} = (q_1, q_2, ..., q_m)$ be a vector of different quotas, $\boldsymbol{\pi}^k$ be a relative power index for quota q_k, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_m)$ be a probability distribution over elements of $\mathbf{q}$. The *randomized voting rule* $(\mathbf{q},\boldsymbol{\lambda})$ selects within different voting acts by random mechanism quotas from $\mathbf{q}$ by the probability distribution $\boldsymbol{\lambda}$. Then $[N,\mathbf{w}]$ has a property of *strict proportional expected power* with respect to a relative power index $\boldsymbol{\pi}$, if there exists a randomized voting rule $(\mathbf{q}^*,\boldsymbol{\lambda}^*)$ such that the vector of the mathematical expectations of power is equal to the vector of voting weights:

$$\boldsymbol{\pi}(N,(\mathbf{q},\boldsymbol{\lambda}),\mathbf{w}) = \sum_{k=1}^{m} \lambda_k \boldsymbol{\pi}^k(N,q_k,\mathbf{w}) = \mathbf{w}$$

The concept of randomized voting rule and strict proportional expected power was introduced by Holler (1982, 1985), and studied by Berg and Holler (1986).

Definition 3 Let $\mathbf{w} = (w_1, w_2, ..., w_n)$ be a fair distribution of voting weights (with whatever principle is used to justify it) in a simple weighted committee $[N,\mathbf{w}]$, $\boldsymbol{\pi}$ is a relative power index, ($\boldsymbol{\pi}[N,q,\mathbf{w}]$ is a vector valued function of q), and d is a distance function, then the voting rule q_1 is said to be *at least as fair* as voting rule q_2 with respect to the selected $\boldsymbol{\pi}$ if $d(\mathbf{w},\boldsymbol{\pi}(N,q_1,\mathbf{w})) \le d(\mathbf{w},\boldsymbol{\pi}(N,q_2,\mathbf{w}))$.

Intuitively, given $\mathbf{w}$, the voting rule q_1 is preferred to voting rule q_2 if q_1 generates a distribution of power closer to the distribution of weights than q_2.

Definition 4 The voting rule q^* that minimizes a distance d between $\boldsymbol{\pi}[N,q,\mathbf{w}]$ and $\mathbf{w}$ is called an optimal voting rule (*optimal quota)* with respect to the selected power index $\boldsymbol{\pi}$.

Let $[N,q,\mathbf{w}]$ be a simple weighted quota committee and C_{is} be the set of critical winning coalitions of the size s in which i has a swing, then

$$\mathrm{card}(P_i) = \sum_{s \in N} \mathrm{card}(C_{is})(s-1)!(n-s)!$$

is the number of permutations with the pivotal position of i in $[N,q,\mathbf{w}]$. The number of pivotal positions corresponds to the number and structure of swings. If in two different committees sets of swing coalitions are identical, then the sets of pivotal positions are also the same.

Proposition 1 Let $[N,q_1,\mathbf{w}]$ and $[N,q_2,\mathbf{w}]$, $q_1 \neq q_2$, be two simple quota-weighted committees such that $W[N,q_1,\mathbf{w}] = W[N,q_2,\mathbf{w}]$, then

$$C_i(N,q_1,\mathbf{w}) = C_i(N,q_2,\mathbf{w})$$

$$P_i(N,q_1,\mathbf{w}) = P_i(N,q_2,\mathbf{w})$$

$$M_i(N,q_1,\mathbf{w}) = M_i(N,q_2,\mathbf{w})$$

for all $i \in N$.

From Proposition 1 it follows that in two different committees with the same set of members, the same weights and the same sets of winning coalitions, the PB-power indices, SS-power indices and HP-power indices are the same in both committees, independently of quotas. Moreover, since the J-index is based on the concept of swing and the DP power index is based on membership in minimal winning coalitions, the J and DP indices are also the same.

Proposition 2 Let $[N,q,\mathbf{w}]$ be a simple quota weighted committee with a quota q,

$$\mu^+(q) = \min_{S \in W[N,q,w]} (w(S) - q) \quad \text{and}$$

$$\mu^{-}(q) = \min_{S \in 2^N \setminus W(N,q,w)} (q - w(S))$$

Then for any particular quota q we have $W[N,q,\mathbf{w}] = W[N,\boldsymbol{\gamma},\mathbf{w}]$ for all $\gamma \in (q - \mu^{-}(q),\ q + \mu^{+}(q)]$.

Proof

a) Let $S \in W[N,q,\mathbf{w}]$, then from the definition of $\mu^{+}(q)$

$$w(S) - q \geq \mu^{+}(q) \geq 0 \Rightarrow w(S) - q - \mu^{+}(q) \geq 0 \Rightarrow S \in W(N,q+\mu^{+},w)$$

hence S is winning for quota $q + \mu^{+}(q)$. If S is winning for $q + \mu^{+}(q)$, then it is winning for any quota $\gamma \leq q + \mu^{+}(q)$.

b) Let $S \in 2^N \setminus W[N,q,\mathbf{w}]$, then from the definition of $\mu^{-}(q)$

$$q - w(S) \geq \mu^{-}(q) \geq 0 \Rightarrow q - \mu^{-}(q) - w(s) \geq 0 \Rightarrow S \in 2^N \setminus W(N,q - \mu^{-}(q),w)$$

hence S is losing for quota $q - \mu^{-}(q)$. If S is losing for $q - \mu^{-}(q)$, then it is losing for any quota $\gamma \geq q - \mu^{-}(q)$.

From (a) and (b) it follows that for any $\gamma \in (q - \mu^{-}(q),\ q+\mu^{+}(q)]$

$$S \in W(N,q,\mathbf{w})\} \Rightarrow S \in W(N,\gamma,\mathbf{w})\}$$

$$S \in \{2^N \setminus W(N,\gamma,\mathbf{w})\} \Rightarrow S \in \{2^N \setminus W(N,q,\mathbf{w})\}$$

which implies that $W(N,q,\mathbf{w}) \Rightarrow W(N,\gamma,\mathbf{w})$.

From Propositions 1 and 2 it follows that swing, pivot and MWC-based power indices are the same for all quotas $\gamma \in (q - \mu^{-}(q),\ q + \mu^{+}(q)]$. Therefore the interval of quotas $(q - \mu^{-}(q),\ q + \mu^{+}(q)]$ we call an *interval of stable power* for quota q. Quota $\gamma^{*} \in (q - \mu^{-}(q),\ q + \mu^{+}(q)]$ is called the marginal quota for q if $\mu^{+}(\gamma^{*}) = 0$.

Now we define a partition of the power set 2^N into equal weight classes $\Omega_0, \Omega_1, ..., \Omega_r$ (such that the weight of different coalitions from the same class is the same and the weights of different coalitions from different classes are different). For the completeness set $w(\emptyset) = 0$. Consider the weight-increasing ordering of equal weight classes $\Omega^{(0)}, \Omega^{(1)}, ..., \Omega^{(r)}$ such that for any $t < k$ and $S \in \Omega^{(t)}$, $R \in \Omega^{(k)}$ it holds that $w(S) < w(R)$. Denote $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1, 2, ..., r$.

Proposition 3 Let $\Omega^{(0)},\Omega^{(1)},...,\Omega^{(r)}$ be the weight-increasing ordering of the equal weight partition of 2^N. Set $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1,2,...,r$. Then there is a finite number $r \le 2^n - 1$ of marginal quotas q_t and corresponding intervals of stable power $(q_{t-1}, q_t]$ such that $W[N,q_t,\mathbf{w}] \subset W[N,q_{t-1},\mathbf{w}]$.

Proof follows from the fact that card $(2^N) = 2^n$ and an increasing series of k real numbers $a_1,...,a_k$ subdivides interval $(a_1,a_k]$ into $k-1$ segments. An analysis of voting power as a function of the quota (given voting weights) can be substituted by an analysis of voting power in a finite number of marginal quotas.

Proposition 4 Let $(N,q,w]$ be a simple quota weighted committee and $(q_{t-1},q_t]$ is the interval of stable power for quota q. Then for any $\gamma = 1 - q_t + \varepsilon$, where $\varepsilon \in (0, q_t - q_{t-1}]$ and for all $i \in N$

$$\text{card}(C_i(N,q,\mathbf{w})) = \text{card}(C_i(N,\gamma,\mathbf{w}))$$

$$\text{card}(P_i(N,q,\mathbf{w})) = \text{card}(P_i(N,\gamma,\mathbf{w}))$$

Proof Let S be a winning coalition, k has the swing in S and $(q_{t-1},q_t]$ is an interval of stable power for q. Then it is easy to show that $N \setminus S \cup k$ is a winning coalition, k has a swing in $N \setminus S \cup k$ and $(1-q_t, 1-q_{t-1}]$ is an interval of stable power for any quota $\gamma = 1 - q_t + \varepsilon \, (0 < \varepsilon \le q_t - q_{t-1})$. Let R be a winning coalition, j has a swing in R, and $(1-q_t, 1-q_{t-1}]$ is an interval of stable power for quota $\gamma = 1 - q_t + \varepsilon \, (0 < \varepsilon \le q_t - q_{t-1})$. Then $N \setminus R \cup j$ is a winning coalition, j has a swing in $N \setminus R \cup j$ and $(q_{t-1},q_t]$ is an interval of stable power for any quota $q = q_{t-1} + \tau$ where $0 < \tau \le q_t - q_{t-1}$.

While in $[N,q,\mathbf{w}]$ the quota q means the total weight necessary to pass a proposal (and therefore we can call it a *winning quota*), the *blocking quota* means the total weight necessary to block a proposal. If q is a winning quota and $(q_{t-1},q_t]$ is a quota interval of stable power for q, then any voting quota $1 - q_{t-1} + \varepsilon$ (where $0 < \varepsilon \le q_t - q_{t-1}$) is a blocking quota. From Proposition 4 it follows that the blocking power of the committee members, measured by swing and pivot-based power indices, is equal to their voting power. It is easy to show that voting power and blocking power might not be the same for power indices based on membership in minimal winning coalitions (i.e. HP and DP power indices). Let r be the number of marginal quotas, then from Proposition 4 it follows that for power indices based on swings and pivots the number of majority power indices does not exceed int $(r/2)+1$.

Proposition 5 Let $q_1, q_2, ..., q_m$ be the set of all majority marginal quotas in a simple weighted committee $[N, \mathbf{w}]$, and $\boldsymbol{\pi}^k$ be a vector of Shapley-Shubik relative power indices corresponding to a marginal quota q_k, then there exists a vector $(\lambda_1, \lambda_2, ..., \lambda_r)$ such that:

$$\sum_{k=1}^{m} \lambda_k = 1,\ \lambda_k \geq 0,\ \sum_{k=1}^{m} \lambda_k \boldsymbol{\pi}^k = \mathbf{w}$$

The proof follows from Berg and Holler (1986). They provide the following property of simple weighted committees: Let $[N, Q, \mathbf{w}]$ be a finite family of simple quota weighted committees with the same weights $\mathbf{w}$ and a finite set of different relative quotas $Q = \{q_1, q_2, ..., q_m\}$. Let $\lambda(Q)$ be a probability distribution over Q where φ_k is a probability with which a random mechanism selects the quota q_k and $\pi_{ik}(N, q_k, \mathbf{w})$ be SS relative power index in the committee $[N, q_k, \mathbf{w}]$ with a quota $q_k \in Q$, then

$$\overline{\pi}_i(N, Q, \mathbf{w}) = \sum_{k: q_k \in Q} \pi_{ik}(N, q_k, \mathbf{w}) \lambda_k$$

is an expected SS relative power of the member i in the randomized committee $[N, \boldsymbol{\lambda}(Q), \mathbf{w}]$. For any vector of weights there exist a finite set Q of quotas q_k such that $0.5 < q_k \leq 1$, and a probability distribution $\boldsymbol{\lambda}$ such that

$$\overline{\pi}_i(N, Q, \mathbf{w}) = \sum_{k: q_k \in Q} \pi_{ik}(N, q_k, \mathbf{w}) \lambda_k = w_i$$

The randomized voting rule $\boldsymbol{\lambda}(Q)$ leads to strict proportional expected SS power. Clearly, if there exists an exact quota q^* such that $\pi_i[N, q^*, \mathbf{w}] = w_i$, we can find it among finite number of marginal majority quotas.□

In general, the number of majority power indices can be greater than the number of committee members, and the system

$$\sum_{k=1}^{r} \lambda_k = 1,\ \lambda_k \geq 0,\ \sum_{k=1}^{r} \lambda_k \boldsymbol{\pi}^k = \mathbf{w}$$

might not have the unique solution. To solve the system we can use the optimization problem:

$$\text{minimize } \sum_{i=1}^{n} abs\left(\sum_{k=1}^{r} \pi_i^k \lambda_k - w_i\right)$$

$$\text{subject to } \sum_{k=1}^{r} \lambda_k = 1, \ \lambda_k \geq 0$$

that can be transformed into an equivalent linear programming problem (Gale, 1960):

$$\text{minimize } \sum_{i=1}^{n} y_i$$

$$\text{subject to } \sum_{k=1}^{r} \pi_i^k \lambda_k - y_i \leq w_i \quad \text{for } i=1,\dots,n$$

$$\sum_{k=1}^{r} \pi_i^k \lambda_k + y_i \leq w_i \quad \text{for } i=1,\dots,n$$

$$\sum_{k=1}^{r} \lambda_k = 1$$

$$\lambda_k, y_i \geq 0 \quad \text{for } k = 1,\dots,r, \ i = 1,\dots,n$$

This problem is easy to solve by standard linear programming simplex methods. Although we can apply a randomized voting rule to any relative power index, based on pivots and swings, the problem is with the interpretation of what we get. The relative PB index has no probabilistic interpretation, so the randomized voting rule calculated for it by Proposition 5 does not provide the mathematical expectation of the number of swings, leading to a relative PB power equal to weights.

One can hardly expect that randomized voting rules leading to the strict proportional expectation of power would be adopted by actors in real voting systems. However, the design of a "fair" voting system can be based on an approximation provided by the quota generating the minimal distance between vectors of power indices and weights, which is called an *optimal quota*.

The optimal quota was introduced by Słomczyński and Życzkowski (2006, 2007) as a quota minimizing the sum of square residuals between the power indices and the voting weights by $q \in (0.5, 1]$

$$\sigma^2(q) = \sum_{i \in N} (\pi_i[N,q,\mathbf{w}]) - w_i)^2$$

Słomczyński and Życzkowski introduced the optimal quota concept within the framework of the so-called Penrose voting system as a principle of fairness in the EU Council of Ministers voting. Here power is measured by the Penrose-Banzhaf power index. The system consists of two rules (Słomczyński and Życzkowski 2007: 393):

a) The voting weight attributed to each member of the voting body of size n is proportional to the square root of the population he or she represents;

b) The decision of the voting body is taken if the sum of the weights of members supporting it is not less than the optimal quota.

Looking for a quota providing a priori voting power "as close as possible" to the normalized voting weights, Słomczyński and Życzkowski (2007) are minimizing the sum of square residuals between the power indices and voting power for $q \in (0.5, 1]$. Based on a simulation they propose heuristic approximations of the solution for the PB index:

$$\underline{q} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{n}}\right) \le q \le \frac{1}{2}\left(1 + \sum_{i \in N} w_i^2\right) = \overline{q}$$

Clearly $\underline{q} = \overline{q}$ if and only if all the weights are equal, but in this case any majority quota is optimal.

Definition 5 By the index of the fairness of a voting rule q in $[N,q,\mathbf{w}]$ we call:

$$\varphi(N,q,\mathbf{w}) = 1 - \sqrt{\frac{1}{2}\sum_i (\pi_i[N,q,\mathbf{w}] - w_i)^2}$$

It is easy to see that

$$0 \le \sqrt{\frac{1}{2}\sum_i (\pi_i[N,q,\mathbf{w}] - w_i)^2} \le 1$$

(zero in the case of the equality of weights and power, e.g. $w_1 = 1/2,\ w_2 = 1/2,\ \pi_1 = 1/2,\ \pi_2 = 1/2$, and 1 in the case of an extreme inequality of weights and power, e.g. $w_1 = 1,\ w_2 = 0,\ \pi_1 = 0,\ \pi_2 = 1$), hence $0 \le \varphi(N,q,w) \le 1$. We say that a voting rule q_1 is "at least as fair" as a voting rule q_2 if $\varphi(N,q_1,\mathbf{w}) \ge \varphi(N,q_2,\mathbf{w})$.[5]

Looking for a "fair" voting rule we can maximize φ , which is the same as to minimize $\sigma^2(q)$. Using marginal quotas and intervals of stable power we do not need any simulation.

Proposition 6 Let $[N,q,\mathbf{w}]$ be a simple quota-weighted committee and $\pi_i(N,q_t,w)$ be relative power indices for marginal quotas q_t , and q_{t^*} be the majority marginal quota minimizing

$$\sum_{i \in N} (\pi_i[N,q_j,\mathbf{w}] - w_i)^2$$

($j = 1,2,...,r$, r is the number of intervals of stable power such that q_j are marginal majority quotas), then the exact solution of Słomczyński and Życzkowski's optimal quota (SZ optimal quota) problem for a particular power index used is any $\gamma \in (q_{t^*-1}, q_{t^*})$ from the quota interval of stable power for q_{t^*} .

The proof follows from the finite number of quota intervals of stable power (Proposition 4). The quota q_{t^*} provides the best approximation of strict proportional power that is related neither to a particular power measure nor to a specific principle of fairness.

4. Concluding remarks

In simple quota weighted committees with a fixed number of members and voting weights there exists a finite number r of different quota intervals of stable power $(r \le 2^n - 1)$ generating a finite number of power indices vectors. For power indices with a voting power equal to blocking power the number of different power indices vectors corresponding to majority quotas is equal at most to int $(r/2) + 1$.

If the fair distribution of voting weights is defined, then the fair distribution of voting power is achieved by the quota that maximizes the index

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[5] The index of fairness follows the same logic as measures of deviation from proportionality used in political science to evaluate the difference between results of an election and the composition of an elected body - e.g. Loosemore and Hanby (1971) is based on the absolute values of the deviation metric, or Gallagher (1991) using a square roots metric.

of fairness (minimizes the distance between relative voting weights and relative voting power). The index of fairness is not a monotonic function of the quota.

The problem of optimal quota has an exact solution via the finite number of majority marginal quotas. Słomczyński and Życzkowski introduced an optimal quota concept within the framework of the so called Penrose voting system as a principle of fairness in the EU Council of Ministers voting and related it exclusively to the Penrose-Banzhaf power index and the square root rule. However, the fairness in voting systems and approximation of strict proportional power is not exclusively related to the Penrose square-root rule and the Penrose-Banzhaf definition of power, as it is usually done in discussions about EU voting rules. In this paper it is treated in a more general setting as a property of any simple quota weighted committee and any well-defined power measure. Fairness and its approximation by optimal quota are not specific properties of the Penrose-Banzhaf power index and square root rule.

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