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Faculty of Social Sciences
Institute of Economic Studies

MASTER THESIS

Global Games

Author: **Bc. Tomáš Fiala**

Supervisor: **PhDr. Martin Gregor, Ph.D.**

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently using only the listed resources and literature and that he has not used this thesis to obtain another degree.

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Prague, July 31, 2012

Signature

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Abstract

In this thesis we review literature about the coordination problem under an uncertainty. We set up a continuum player model of collective action, in which part of the population must coordinate on an action in order to achieve a mutual benefit. The complete information version of the model features multiple equilibria. We study the role of various sources of uncertainty in the model and compare them. We also examine the role of private and public information.

We discuss particularly the global game, the coordination game of incomplete information in which agents received different but correlated signals about the state. We demonstrate that in the global game an unique equilibrium can be found by iterated elimination of dominated strategies. We compare the global game to related models and examine the consequences of relaxing the assumptions of global game.

In addition we show some practical implication of the model for revolutions and currency crises.

Keywords: Global game, coordination problem, collective action, higher order beliefs, revolutions, currency crisis

Abstrakt

Tato diplomová práce se zabývá koordinačním problémem za nejistoty. Zabýváme se modelem koordinační hry s kontinuem hráčů, ve kterém určitá část hráčů musí zvolit stejnou akci, aby docílila změny režimu. Verze tohoto modelu s dokonalou informovaností má dvě Nashovy rovnováhy. Zavádíme do tohoto modelu nedokonalou informovanost a porovnáváme důsledky nejistoty o různých parametrech tohoto modelu. Dále zkoumáme roli privátních a společných informací.

Především se zabýváme tak zvanou globální hrou, verzí tohoto modelu s nedokonalou informovaností, ve které hráči pozorují rozdílné, ale korelované signály. Ukážeme, že v tomto případě má model jedinou Nashovu rovnováhu. Dále se zabýváme podobnými modely a na jejich příkladě ukážeme klíčové vlastnosti, které způsobí jednoznačnost rovnováhy v globální hře.

Nakonec ukážeme některé aplikace tohoto modelu na revoluce a měnové krize.

Klíčová slova: Globální hry, koordinační problém, kolektivní akce, nejistota vyššího řádu, revoluce, měnové krize

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Chapter 1

Introduction

In many situations, group of people can reach mutual gain only if they manage to coordinate. One of such situation is a revolution. When citizens disagree with a government and want to overthrow it, they must act against the government in coordinated fashion. The government can defend small revolt, but if all people participate on the revolution, he has to give up. The people are willing to take the actions only if everyone else participates as well.

The need for a coordination also arises in many economic settings. Take for example a speculative attack against a currency. When large number of investors speculate on a depreciation of a pegged currency, they short sell large amount of the currency, force the central bank to abandon the fixed exchange rate regime and the investors earn money from the speculation. They can earn the gain, however, only if large number of the investors short sells the currency simultaneously. Other examples include debt crises, bank runs or network externalities.

The results of such situations are self-fulfilling. The beliefs of the people determine the outcome. If the investors believe that the bank will abandon the peg, they will speculate on the depreciation of the currency and eventually force the bank to depreciate. If, however, the investors believe that the bank will maintain the fixed exchange rate regime, nobody participates on the speculation and the bank is able to peg the currency. There are multiple coherent beliefs and based on the theory of Nash equilibria, we cannot predict the outcomes.

In the real world, however, neither the people nor the outsider can be exactly sure about the all aspects of the situation. It has been long tradition in the game theory to examine a slightly perturbed model. Interestingly the perturbation sometimes leads to an unique prediction.

In this thesis we will examine such perturbed model of a situation that requires coordination. We will examine the coordination game under uncertainty, in which the players receive idiosyncratic but correlated signals about the game. Carlsson & van Damme (1993) call such a perturbed game a global game, because it assumes that the players take in account large class of games rather then one generally known game. There are two advantages to this approach. For one thing, it can often provide unique prediction in the coordination game and for the other, it gives us easy framework how to model the impact of information.

In this thesis we survey the theoretical literature on the global game and related approaches. We introduce various models that appeared in literature and introduce them under a common framework of a model of collective action. We look at a complete information model of collective action and then consider the effect of various sources of uncertainty. We compare models that are global games and that are not and illustrate on them the key properties that drive the result of uniqueness in the global game.

In the next chapter we introduce the coordination problem in the simplest setting of 2×2 coordination game. We review approaches to equilibrium selection in the coordination game. We introduce the global game and comment on the intuition behind the uniqueness of equilibrium.

In the third chapter we introduce a continuum player counterpart of the 2×2 coordination game. In the model, a sufficient proportion of the population must coordinate on an action to achieve a change of regime. We introduce uncertainty about the strength of the regime into the model. We investigate the model under various information structure. In fourth chapter we look at consequences of uncertainty that stems from heterogeneity in the benefits from the change of the regime.

In the fifth chapter we show two practical implications of the model. We show that the model gives rationale of terrorist action as a way to communicate information and we apply the findings of our model to a model of currency crisis. The last chapter concludes.

Chapter 2

Coordination Game

The goal of this chapter is to outline a problem of coordination and show that beliefs and informations play a vital role in it. We present a simple game that illustrates the key aspects of coordination. We interpret the game as an attempt to overthrow a leader or government, although the model is far more general and permits other interpretations as well.

We suppose that there are two players who want to overthrow a leader. They decide simultaneously whether to revolt against the leader or abstain. The leader is strong and can withstand one man revolt. Player who revolts just by himself does not have sufficient power to threaten the current regime. For the revolution to be successful both of the players must revolt. Each of the players gets a positive reward of utility B, when the regime is overthrown. The leader punishes the unsuccessful revolution by lowering the utility of the revolting player to $-c$, where $c > 0$. If a player abstains he risks nothing and receives payoff 0 independently of what the other player does. We summarize the payoffs of the game in the table 2.1.

This game is a classical example of a coordination game (also called assurance game or a stag hunt game). It describes a situation in which players must coordinate on some action (revolting) to achieve a Pareto superior outcome. The unsuccessful attempt to coordinate is costly, however, so the players must decide between taking the safe action that gives sure payoff or trying to reach

	revolt	abstain
revolt	B,B	$-c$,0
abstain	0,$-c$	0,0

Table 2.1: Payoff Matrix of the Coordination Game

the highest payoffs. When he decides to get the highest benefit, he relies on the other player that he chooses the same action.

The game features two pure Nash equilibria. In one Nash equilibrium both players coordinate, in the second both abstain. In addition there is a mixed Nash equilibrium. The multiplicity of equilibria gives rise to the indeterminacy of the outcome. An outside observer cannot predict what happens solely on the basis of a Nash equilibrium . Also the players themselves cannot be sure about the actions of others. In such game each player wants to choose the same action as the other player and the outcomes are self-fulling. If a player expects the other one to revolt, he will also revolt. Every outcome is possible and it depends only on the expectations of the players which one will be realized. Such situation poses problem both for an observer who cannot make predictions and for the players who must select their action amid uncertainty about the choices of others. Hence we dub such circumstances as coordination problem.

While the indeterminate nature of coordination game poses a difficulty for a prediction, it can also explain why an outcome of a game can change unexpectedly without any change in the underlying parameters of the game. The change in expectations about other players induces a change in the outcome even if the payoffs themselves stay the same. This trait is often seen during revolutions, but there many other economic or political examples. Shiller (2000) describes how self-fulfilling prophecies of the type of coordination game give rise to bubbles and crashes in financial markets. Cooper *et al.* (1992) surveys the role of self-fulfilling prophecies in macroeconomics. We will give some examples of important economic situations in which such coordination problem arise in chapter 5. In this chapter we will survey theoretical approaches that attempted to identify the actions that the players would be most likely to select.

2.1 Equilibrium Selection

2.1.1 Instability of a Mixed equilibrium

We already noted that in the game there are two pure strategy Nash equilibria and that there is also a mixed equilibrium. This equilibrium, however, appear not to be a good prediction of actual behaviour. In this section we argue for the exclusion of the mixed equilibrium.

First, we characterize the mixed equilibrium. We describe the strategies of player one and two by p_1 and p_2, the probabilities with which players one and two revolt. Player one revolts if his expected utility from revolting is greater than from abstaining. Therefore if:

$$p_2 B - (1 - p_2)c \geq 0$$

Solving for p_2 gives us condition that player one revolts if:

$$p_2 \geq \frac{c}{B + c}$$

Therefore player one revolts if player two revolts with high probability. By symmetry of the game, we obtain that player two revolts only for high p_1 as well. Thus, the player one (two) revolts whenever the probability of revolting of player two (one) is higher then threshold $\overline{p}_2$ ($\overline{p}_1$), where the thresholds are given as:

$$\overline{p}_1 = \overline{p}_2 = \frac{c}{B + c}$$

We can summarize the best response correspondences in a standard best response graph in figure 2.1. The best responses are represented by solid thick lines and the intersections of the best responses form Nash equilibria. We denote them by black circles. Thus revolting with probability $\overline{p}_1$ is the mixed equilibrium strategy for both players.

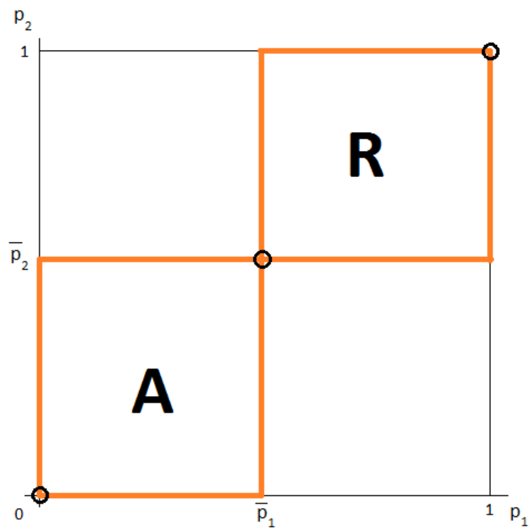

Figure 2.1: The Best Response Graph

Echenique & Edlin (2004) argued that the mixed equilibrium in our setting is unstable. For the mixed strategy to be an equilibrium each player must believe with that the other revolt exact probability $\overline{p}_1$. Even a infinitesimal change in beliefs induces a player to adopt a pure strategy. Suppose that we perturb the beliefs downwards, then both players play the pure strategy abstain, which in turn forms the new equilibrium.

After any perturbation of the mixed equilibrium beliefs the best response takes the strategies away from the mixed equilibrium. In addition Echenique & Edlin (2004) argue that when players play mixed strategies, then after several plays in which player repeatedly abstained, the players might revise their beliefs downwards and hence switch to pure strategy equilibrium. Echenique & Edlin (2004) show that the argument holds for a broad class of learning dynamics. The mixed equilibrium does not seem to be a sensible prediction and in what follows we will be concerned only with the choice between pure strategy equilibria.

2.1.2 Focal Point

Schelling (1980) asserts that in many coordination problems there is a natural choice of behaviour. Imagine an example of man and wife lost in the city. The couple got lost in a foreign city and they do not know where is the other one. Both desire to meet the other, but they cannot communicate. Hence they need to choose a place where they expect the other one to be. The choice of the place where to meet poses a coordination problem similar the one outlined above. Although there is infinite number of possibilities where to wait, some places appear more natural than others. In such situation one would most likely choose to meet at a railway station, main square or other important place. Schelling (1980) posed similar question to a sample of people from New Heaven. He asked them where and when they would meet with other person in New York. Majority of them answered that they would meet at information booth at Central Railway Station and virtually all of them at 12 noon.

Schelling (1980) gives other examples of coordination problems, where people manage to coordinate because they prefer for whatever reason one particular action over another. This indicates that experience with similar coordination problems has led to establish some actions that are expected to be played by others. Such a collection of actions that are likely to be selected forms a focal point.

The players can also cooperate on exogenous signals. Some publicly observed signal can form expectation that other will revolt and thus induce me to revolt and further strengthen the motivation of the other player to participate on the revolt. Such outcome changing signal could be irrelevant to the setting of the game in terms of payoffs and yet change the decisions of the people. This argument gives rationale to the role of various leaders. By creating an action that is publicly visible, they motivate other players to change their behaviour.

2.1.3 Payoff and Risk Dominance

In an attempt to select unique equilibrium, Harsanyi & Selten (1988) formulated two concepts for choice between multiple equilibria in a 2×2 game: payoff and risk dominance.

An equilibrium is payoff dominant if it yields strictly higher payoffs for all players than any other equilibrium. Harsanyi & Selten (1988) argue that if there is payoff dominant equilibrium the players should select it. Their argument relies on the possibility of pre-play communication. If the pre-play communication is allowed and players can make an agreement on which actions they will play, they should agree to play a Pareto efficient equilibrium. The agreement should be on equilibrium actions, because such agreement would be self-enforcing and players would trust that the other would honour the agreement and it should be Pareto efficient, because if the equilibrium was not Pareto efficient, they could change their agreement and make everyone better off.

The argument above supposes pre-play negotiation, but even without the communication the payoff dominant equilibrium is natural choice as a focal point. Since both players know that if they could communicate they would agree on the payoff dominant equilibrium, they might as well select it without communication on the basis of a tacit agreement. Without any additional information it might be reasonable to assume that opponent chooses the action leading to payoff dominant equilibrium. In our example in the payoff dominant equilibrium both players revolt. So the discussion would suggest that we will always see payoff dominant equilibrium being played, yet, in many cases we witness coordination failures (we survey experiments that confirm it in subsection 2.1.5).

The payoff dominance approach bases the selection of the equilibrium only on the payoffs that are achieved in equilibrium, however, The concept does not take in the account the consequences of when the other player fails to play equilibrium action. When the costs of the failed revolt are very high, even slight grain of uncertainty about the action of the other player could decrease the expected gain from revolting substantially. Without pre-play communication, the players can never be entirely certain that other will choose the payoff dominating equilibrium action. Thus, players might rather select safe action instead of the action that promises highest payoffs. For the 2×2 games, Harsanyi & Selten (1988) gave precise definition what it means that an action is safe.

Let there be a general 2×2 game. We will name the actions as in our example. We assume that it has also three Nash equilibria (*revolt*, *revolt*; *abstain*, *abstain* and a mixed one). We describe the strategies as in the previous section as probability of revolting (p_1 and p_2) and the $\overline{p}_1$ and $\overline{p}_2$ will denote the threshold probabilities. Thus the best responses are to revolt if $p_2 \geq \overline{p}_2$, respectively if $p_1 \geq \overline{p}_1$. We will say that equilibrium *revolt*, *revolt* is risk dominant if $\overline{p}_1 + \overline{p}_2 < 1$. If the game is symmetric the condition reduces to $\overline{p}_1 = \overline{p}_2 < \frac{1}{2}$.

2.1.4 Rationale for Risk Dominance

Harsanyi & Selten (1988) give the rationale for the concept of risk dominant equilibria. They showed that the risk dominance is the only method that selects between two equilibria in 2×2 games that satisfy certain axioms. [1] Below we present different justifications for risk dominance that appeal to the notion that risk dominant equilibrium strategies are safer.

The best response diagram on figure 2.1 id divided in four quadrants. The definition of risk dominance of equilibrium *revolt*, *revolt* is equivalent to a situation, when the area of the upper right quadrant denoted by R is greater than the area of the lower left quadrant denoted by A. To see that, the area of A on left hand side of equation is greater than the area of R in on the right hand side when:

$$\overline{p}_1\overline{p}_2 > (1 - \overline{p}_1)(1 - \overline{p}_2)$$

and that is equivalent to the condition for risk dominance of the (*revolt*, *revolt*) equilibrium:

$$\overline{p}_1 + \overline{p}_2 < 1$$

The best responses of both players to the beliefs in upper right quadrant

[1] These axioms are: invariance with respect to isomorphism (the risk dominance selects the equilibrium independently on renaming or reordering the strategies), best reply invariance (risk dominance depends only on best reply structures) and payoff monotonicity (after raising the payoffs for the risk dominant equilibrium outcome the equilibrium is still risk dominant).

lead to equilibrium, where both revolts and similarly beliefs from the lower left quadrant lead both players to play the equilibrium actions $abstain, abstain$. Hence, the actions played in risk dominant equilibrium are supported by more beliefs than actions in the other equilibrium. Thus risk dominant equilibrium may be perceived as safer, because higher number of beliefs lead to it. In addition, Young (1993) shown that broad class of stochastic learning processes converge to the risk dominant equilibrium.

Harsanyi & Selten (1988) offer another heuristic justification for the risk dominance criterion. Consider that player two is uncertain about the belief of player one about the probability he revolts. Therefore he is uncertain about what value player one assigns to p_2. By the principle of insufficient reason, he thinks that player one beliefs in each value of p_2 equally likely, i.e. player two has uniform prior over the beliefs p_2 of player one. Player one revolts if $p_2 > \overline{p}_2$. Thus the player two thinks that player one will revolt with probability $\Pr(p_2 > \overline{p}_2)$. Because he has the uniform prior about p_2 he thinks that player one revolts with probability $p_1 = 1 - \overline{p}_2 = \Pr(p_2 > \overline{p}_2)$. The player 2 will revolt if $p_1 > \overline{p}_1$. Hence the equilibrium $revolt, revolt$ is justifiable for player 2 by this stream of reasoning whenever $1 - \overline{p}_2 = p_1 > \overline{p}_1$, thus whenever it is risk dominant. The same condition is obtained if we switch player two for player one. In this case, the risk dominant equilibrium is a result of the uniform priors over the beliefs of other players.

It is remarkable that several arguments, albeit they seem ad hoc and unrelated, support the selection of risk dominant equilibrium. Under the presence of strategic uncertainty (the uncertainty about the moves of other players rather then the uncertainty about the state of the world) the risk dominant equilibrium seems to be the safer option. The next subsections confirms that the risk dominant equilibria are often selected in experimental settings.

2.1.5 Experimental Evidence

In our example the equilibrium where both players revolt is payoff dominant. If the parameters are such that $\overline{p}_1 = \frac{c}{B+c} < \frac{1}{2}$, however, the equilibrium $revolt, revolt$ is not risk dominant. This means that there is often trade-off between the payoff and risk dominance criteria. If the benefit from successful revolt is small relative to the punishment, the Pareto efficient outcome does not coincide with the risk dominant equilibrium. Harsanyi & Selten (1988) favour the criterion of payoff dominance over the risk dominance on the ground that

	revolt	abstain
revolt	1000,1000	0,800
abstain	800,0	800,800

Table 2.2: The Payoffs in the Experiment of Cooper *et al.* (1992)

if players communicate they would agree on Pareto efficient equilibrium. If, however, communication is not allowed the risk dominant outcome might be more likely. Several experimental studies (Schmidt *et al.* 2003; Straub 1995; Cooper *et al.* 1992) suggested that risk dominant equilibrium is indeed better prediction for the actual choices that people make in experimental setting.

We will describe one of the experiments more closely. Cooper *et al.* (1992) tested experimentally, which equilibrium is selected in the coordination game with the payoff matrix given in the table 2.2. We label the actions as *revolt* and *abstain* for comparison, although Cooper *et al.* (1992) used different interpretation and names of actions. The game of the experiment is linearly transformed coordination game of our example with $B = 200$ and $c = 800$. The cost is relatively large compared to the potential benefit and therefore the equilibrium $abstain, abstain$ is risk dominant.

The players were matched randomly with each other several times to play the game. Cooper *et al.* (1992) found that without pre-play communication the players selected the risk dominant equilibrium action *abstain* in more than 98 percent of the cases. When only one player was allowed to send a signal (only one player could send a message *I will abstain* or *I will revolt*), the players choose the action *abstain* in 45 percent of the cases. When both of the players could send signal the action *abstain* was played only in 4 percent of the cases. The result in the first and the last case support the argument, that without pre-play communication players select risk dominant actions while when they can communicate they choose the payoff dominant actions. The case with just one signal is remarkable. Even pre-play communication in the form of one signal does not establish enough certainty that other player will play Pareto efficient action is all cases.

2.2 Global Game

Carlsson & van Damme (1993) gave another rationale for the selection of risk dominant equilibria. They examined the game slightly perturbed with noise. In their model the payoffs of the game depend on random parameters and

	revolt	abstain
revolt	θ,θ	$\theta - \gamma$,0
abstain	0,$\theta - \gamma$	0,0

Table 2.3: Payoff Matrix of the Linear Example

players receive noisy idiosyncratic signals about the parameters. Moreover they assumed that for some states one action is dominant and for some states the second action is dominant. Carlsson & van Damme (1993) showed that as the noise in the signals vanishes, the risk dominant equilibrium will be the only strategy surviving iterated deletion of dominated strategies at each state.

We illustrate their result with an example based on the revolution game we analysed. To make the analysis of the example easy we make other assumptions that might seem unreasonable, yet they serve the purpose of illustrating the argument. We transform the game to the simple linear example used by Morris & Shin (2001)(who modified the example given by Carlsson & van Damme (1993)) and follow their analysis to illustrate the global game.

2.2.1 Linear Example

We suppose there is some underlying one-dimensional state of the world that selects the payoffs of the game. We assume there is a random parameter θ that parametrizes both the reward from the successful revolution and the punishment for the unsuccessful one. The state captures the strength of anti-government sentiment of the players as well as the ability of the government to punish the revolutionaries, so the expected benefit from revolting rises with the state. We assume in particular that the payoffs are affine function of the state in the form:

$$B = \theta, \; c = \gamma - \theta, \; \gamma > 0$$

The payoffs of the transformed game are summarized in the table 2.3. When the state θ is between zero and γ the game has two pure equilibria and there is coordination problem. When θ is lower than zero abstaining is dominant strategy and when θ is higher than γ revolting is dominant strategy. The payoffs are quite artificial, as it is hard to imagine player being rewarded for unsuccessful revolution, they are meant to demonstrate the global game argument, for which it is necessary that both strategies are dominant for some states. In subsequent chapters we will use similar method in more natural setting.

We assume that the parameter θ comes from improper uniform prior and that each player receives an idiosyncratic signal about the state in the form $x_i = \theta + \eta_i$, where η_i are independent and come from normal distribution with zero mean and variance σ. After a player observes his signal, he forms the posterior about the parameter θ. The uniform prior of θ is not a proper probability distribution, nonetheless the posterior distribution of θ given signal x_i is well defined. The posterior distribution of θ conditional on signal x_i is $\mathcal{N}(x_i, \sigma)$.

The signals convey information not only about the underlying state, but also about the other player's signals. This is crucial point that generates the uniqueness of equilibria. In this respect this game differs from the usual game with imperfect information. The player's i belief about signal of player j is distributed according to $\mathcal{N}(x_i, 2\sigma)$.

We will search for a strategy in a special form of a threshold strategy. Thus we will assume that players will revolt if they observe signal higher then threshold $\tilde{x}$. It seems natural that players revolt if they receive high signals about the θ, because the payoff B is increasing and the cost c is decreasing in θ. It turns out that threshold strategy does not only form an equilibrium but that it is also unique equilibrium. [2]

Therefore given that player 2 follows the cutoff strategy, that he revolt whenever his signal is weakly greater than $\tilde{x}$, the expected payoff to revolting of type x_1 of player 1 is:

$$
\begin{aligned}
&\mathbb{E}[\mathbb{I}_{x_2 \geq \tilde{x}}\theta + \mathbb{I}_{x_2<\tilde{x}}(\theta - \gamma) \mid x_1] = \\
&= \mathbb{E}[(1 - \mathbb{I}_{x_2<\tilde{x}})\theta + \mathbb{I}_{x_2<\tilde{x}}(\theta - \gamma) \mid x_1] = \\
&= \mathbb{E}[\theta + \mathbb{I}_{x_2<\tilde{x}}(-\theta + \theta - \gamma) \mid x_1] = \\
&= \mathbb{E}[\theta \mid x_1] - \mathbb{E}[\mathbb{I}_{x_2<\tilde{x}}\gamma \mid x_1] = \\
&= x_1 - \Pr(x_2 < \tilde{x} \mid x_1)\gamma \\
&= x_1 - \Phi\left(\frac{\tilde{x} - x_1}{\sqrt{2\sigma}}\right)\gamma
\end{aligned} \tag{2.1}
$$

Hence, player one revolts if the expected payoff from revolting is weakly greater than zero or if:

[2] Strictly speaking there are actually two strategies, because the player will be indifferent between revolting and abstaining after he receives the threshold signal. Since the probability of him receiving the threshold signal is zero we will ignore it and assume that whenever a player is indifferent he chooses to revolt.

$$x_1 \geq \Phi\left(\frac{\tilde{x} - x_1}{\sqrt{2\sigma}}\right)\gamma \tag{2.2}$$

Since the left hand side of the equation 2.2 is increasing and the right hand side is decreasing in x_1, player one will revolt if he receives high signal x_1. This justifies the assumptions that players play threshold strategies. For the cutoff strategy to be equilibrium, in the threshold signal signal $\tilde{x}$ the expected payoff of revolting must be equal to zero. We plug $\tilde{x}$ for x_1 and get:

$$\tilde{x} - \Phi\left(\frac{\tilde{x} - \tilde{x}}{\sqrt{2\sigma}}\right)\gamma = 0$$

After subtracting the term in the cumulative distribution function we get that the probability of revolting is one half and the value of the threshold signal is:

$$\tilde{x} = \frac{1}{2}\gamma \tag{2.3}$$

2.2.2 Equilibrium Uniqueness

The threshold strategy does not only form an equilibrium, it is also the only strategy that survives the iterated deletion of strictly dominated strategy. The fact that the unique equilibrium can be found by iterated elimination is a consequence of coordination nature of the game. More precisely the global game falls in the category of games with strategic complements, for which unique equilibria are dominance solvable.

We can loosely define the games of strategic complements as games with ordered strategies that have payoffs such that the difference of the benefit from playing higher strategies and the benefit from playing lower strategies is increasing in the strategies of other players. The key property of such games is that the best responses are increasing. These games involve some element of coordination, players want to either cooperate on low strategies or play together the higher strategies. The high strategies mutually reinforce each other. The concept of games of strategic complements generalizes the simple 2×2 coordination game.

Milgrom & Roberts (1990) found that for this class of games one can find lowest and highest equilibria by iterated elimination of dominated strategies. Moreover the iterated elimination proceed by iteratively finding best responses to the lowest and highest strategy. This in turn implies that if there is unique equilibrium, it will be possible to find it by the iterated elimination of dominated strategies. The global game is example of such game, and the elimination of dominated strategies will lead to the unique equilibrium. Below we will illustrate the technique by showing that the equilibrium strategy we derived in last section also forms an unique equilibrium.

We described the strategy of player i in the global game as S_i, the set of all types of player i who revolt. The best response of player i to a strategy S_j is a strategy that maximizes the expected payoff of player i given a strategy S_j of player j. We denote as $BR_i(S_j)$. Further we assumed that whenever player is indifferent he chooses to revolt. Hence after using the formula for expected payoff 2.1, we get that the best response is:

$$BR_i(S_j) = \{x_i : x_i \geq \Pr(x_j \in S_j | x_i)\}$$

The strategies are order by set inclusion. Since $S_j \subseteq S'_j$ implies $Pr(x_j \in S'_j | x_i) \leq Pr(x_j \in S_j | x_i)$. Hence whenever $x_i \in BR_i(S_j)$ it is also $x_i \in BR_i(S'_j)$. So the best response is increasing and we will exploit the fact.

The process of iterated elimination of dominated strategies starts at the smallest strategy. In our case the smallest strategy is a strategy where all types of player do not revolt i.e. S_j is empty. Because all strategies are greater than the minimal strategies and the best response function is increasing, best response to any strategy must be bigger than $BR_i(\emptyset)$. In other words, the types of a player i that are in $BR_i(\emptyset)$ would always revolt, even if nobody else participated. For those types, revolting is dominant strategy. Thus $BR_i(\emptyset)$ is lower bound on strategies after first round of elimination of strictly dominated strategies. Hence all strategies smaller than $BR_i(\emptyset)$ are eliminated.

Similarly after second round of elimination $BR_j(BR_i(\emptyset))$ constitutes the lower bound of all strategies that are best responses of player j to any undominated strategy of player i. The types that are in $BR_j(BR_i(\emptyset))$ are the types for whom it is dominated to revolt even if they play against a strategy, where the minimum of types of player i revolt. And similarly applying best responses over and over to the smallest strategy leads to the smallest undominated strategies after several rounds of elimination. We proceed by the repeated application of the best response to find the smallest strategy surviving iterated elimination of dominated strategies, set of all types for whom it is iteratively dominant to revolt. Then by repeated application of the best response to the highest strategy we find an upper bound on all undominated strategies. In the end we show these two strategies coincide in the case of the global game.

Because the game is symmetric, both players have the same best response functions. We show that a best response to a threshold strategy is again a threshold strategy. So we will characterize the strategies by sequence of thresholds. The best response to a threshold strategy $\langle t, \infty)$ is:

$$BR(\langle t, \infty)) = \{x_i : x_i \geq 1 - \Pr(\langle t, \infty)|x_i)\}$$

Since the left hand side of the inequality is strictly increasing and the right hand side is strictly decreasing in t, the best response is another threshold strategy with a threshold b. We define the function $b(t)$ that gives a threshold in the best response to a strategy with threshold t. So $b(t)$ the value of b that implicitly solves for value t:

$$b = 1 - \Pr(\langle t, \infty)|b)$$

We will denote the repeated application of the function $b(t)$ as $b^n(t)$. Then the lowest strategy, in the above terminology, is the one with $t = \infty$, the strategy where no type revolt. After first round of elimination for all types x greater than $b(\infty)$ it will be dominated to revolt. By application of $b^(t)$ to the greatest strategy $t = \infty$, for all types x smaller than $b(-\infty)$ it will be dominated to abstain.

After the nth round of elimination of the dominated strategies, in the undominated strategy a player will abstain if his type is smaller than $b^n(-\infty)$ and revolt if his type is greater than $b^n(\infty)$. Because these sequences are increasing (respectively decreasing) and bounded, they have limit. For the uniqueness of equilibrium we must show that the limit of $b^n(\infty)$ and $b^n(-\infty)$ is the same.

Because the $b(t)$ is continuous. For the limit of $b^n(\infty)$ it must be that:

$$\lim_{n\to\infty} b^n(\pm\infty) = \lim_{n\to\infty} b^{n+1}(\pm\infty) = b(\lim_{n\to\infty} b^n(\pm\infty))$$

Hence the limit of $b^n(\infty)$ and $b^n(-\infty)$ must be a fixed point of $b(t)$. In the previous subsection we found that there is only one fixed point of $b(t)$. So we found that the elimination of strictly dominated strategies leads to a selection of an unique equilibrium strategy.

2.2.3 General Global Game and Risk Dominance

We found that the unique equilibrium is the strategy for which a player i revolts if for his type x_i:

$$\frac{1}{2}\gamma \leq x_i$$

We look what is relation between this outcome and risk dominance. We repeat that the equilibrium $revolt, revolt$ is risk dominant in the game of perfect information if:

$$\bar{p}_1 = \frac{c}{B+c} < \frac{1}{2} \tag{2.4}$$

After we plug in the values of θ in the condition 2.4 we get that the equilibrium $revolt, revolt$ is risk dominant for the realizations of θ for which:

$$\frac{\gamma - \theta}{\theta + \gamma - \theta} < \frac{1}{2}$$
$$\frac{1}{2}\gamma < \theta$$

The condition for the equilibrium $revolt, revolt$ to be risk dominant is similar to the condition that gives the types that revolt in equilibrium The values of θ where the equilibrium is risk dominant coincide with the values of signals for which the players will revolt. The player will revolt if his signal indicates that the equilibrium $revolt, revolt$ is risk dominant. This outcome is no coincidence and even in very general setting there is connection between risk dominance and the equilibrium of the global game. Carlsson & van Damme (1993) showed that for very general assumptions about prior and signals as noise vanishes the risk dominant equilibrium will be played.

In the general setting Carlsson & van Damme (1993) assumed that there is 2×2 game, of which payoffs depend on a possibly multidimensional random variable (state). Each player receives an unbiased signal about the state. Under other technical assumptions, they showed that if: for some state θ one equilibrium is risk dominant (say $revolt, revolt$), there is continuous curve in the support of the random variable that joins θ and other state where revolting is dominant, and in all states on the curve $revolt, revolt$ is risk dominant, then as the noise vanishes abstaining at θ does not survive the iterated elimination of strictly dominated strategies. In other words this says that under the setting of idiosyncratic perturbations the players will be forced to select the risk dominant equilibrium. The uniqueness of the choice of the strategy will spread from the states where the strategy is dominant to the states it is only risk dominant. Thus Carlsson & van Damme (1993) provide strong foundation for risk dominant criterion.

In the light of the general result, the assumption on the one-dimensionality of the state and the particular form of the function that map the state to payoffs are not necessary to obtain the result. As the noise vanishes the players would revolt whenever the signals would suggest that revolting is risk dominant.

2.3 Beliefs and the Global Game

The result of global game results gives rationale for the risk dominant criterion. The result is quite remarkable, because as the noise vanishes there is almost no uncertainty about the state of the world. With small variance of the signals both players could be fairly certain that they would benefit from the revolution, yet they would both choose to abstain.

Consider an example when $\gamma = 1$, then if player one receives signal of a value $x_1 = 0.45$ and the variance of the signal is 0.0001. He believes with 95 percent confidence that the parameter θ lies in an interval $(0.43, 0.47)$, so he is fairly certain that he would benefit from successful revolution. Not only he knows that he would benefit from the revolution, he also know that player two is fairly certain about the fact. Player one believes that the signal x_2 lies with 95 percent probability in an interval $(0.41, 0.49)$ and hence he believes that the player two is fairly certain that he would benefit from the revolution. Even though the player one is convinced that the revolution would be beneficial for both of the players, he still chooses to abstain.

The reason for this result is that it is not enough for a player to believe that he and the other player would benefit from the revolt. We illustrate the reason why such knowledge is not enough by a story. Schelling (1980) described the important role of knowledge about knowledge in the coordination setting. He gave a story that illustrates that the uncertainty about others intentions matters a lot in coordination. Imagine a situation, when a burglar with a gun is surprised by a homeowner who also happens to have gun. Everyone would rather avoid confrontation and leave the spot. Yet if the other one is determined to shoot, it is better to shoot him first. The homeowner does not shoot if he is fairly certain that the burglar would not shoot, but then burglar must be almost sure that the homeowner will not shoot, so he must believe that the homeowner believes that he will not shoot. Hence the homeowner will believe that the burglar will not shoot if he believes that burglar believes that the homeowner believes that the burglar will not shoot. Thus simple belief that both players have common interest to avoid confrontation will not be enough to avoid it. The players must have strong belief about beliefs about beliefs and so on ad infinitum. As we move to higher order of beliefs about beliefs the variance of the 95 percent confidence interval accumulates. As a result Even small grain of doubt about the state could accumulate into large uncertainty about others' beliefs and actions.

When all players know an event and they also know that others know it and so on, we will say that the event is common knowledge. Aumann (1976) introduced the term. Yet, in our case and so many other situations no player knows for sure what state of the world happened. The only information that player has about the state is his signal, and he does not certainly know the other components of the state. It turns out that common knowledge could be approximated by the similar notion of common belief. Monderer & Samet (1989) introduced the p-beliefs a generalization of the notion knowledge of Aumann (1976).The reason why small uncertainty of the signals matter is the lack of high common beliefs.

Rubinstein (1989) comes up with another example[3] that illustrates the importance of common knowledge. In the example two players can to coordinate on a revolt. There are two states. In the first state the success of the attack depends on whether both players participate and the action revolt is risk dominated. In the other state the revolution is always unsuccessful. Player one observes the state and sends a message to other player and then the other player sends a confirmation that he received the message and then the player one sends a confirmation and so on until the message is lost. Each time they send it there is certain probability that message gets lost. Remarkably for any number of exchanged messages, it will always be dominant not to participate on the revolt, because there will never be common knowledge (or sufficiently high belief) that both will participate. This result reminds the outcome of the experiment of Cooper *et al.* (1992), where even if one player could send signals the players still could not coordinate on risk dominated action.

The global game gives a elegant version of the story of Schelling (1980) and it argues that only risk-dominant equilibria are immune to the lack of the

[3]The story that sets the model is different, we use the setting of our example.

common belief. Below we describe the information structure and use the notion of p-belief to describe the beliefs and the beliefs about beliefs and their effects on the sustainability of equilibria in the linear example.

2.3.1 Information Structure

To describe the effect of beliefs, we must define well the information structure of the game. The information structure of a game consists of a state space Ω, the prior distribution of the states and the partitions of Ω, one for each player that specifies what each player knows about the realized state of world. The state of a world is a complete description of the game. The nature draws the state and players observe that the state belong to some set in their partition. An event E is a set of states, an event happens if state ω is selected and $\omega \in E$.

In our example the state is a triple (θ, x_1, x_2) that consists of the parameter that selects the payoffs and the two signals of the players. The state space is $\mathbb{R}^3$. Each player receives just his signal, therefore after he receives signal of value x_1 his information is that he knows that the event $\{(\Theta, X_1, X_2) : X_1 = x_1\}$ has happened.

2.3.2 Beliefs

Monderer & Samet (1989) define the notion of p-belief. We say that player i p-beliefs event E at a state ω if his posterior probability given the information he has at state ω is equal to or higher than p. More specifically in our example, player 1 p-beliefs event E at state (θ, x_1, x_2) if $\Pr(E|\theta_1) \geq p$. We will denote a set of states where player i p-believes an event E as $B_i^p E$. In our setting this means that $B_i^p E$ is defined as:

$$B_i^p E = \{(\theta, x_1, x_2) : \Pr(E \mid x_i) \geq p\}$$

We say that players have a common p-belief about an event E at ω, when at ω everyone p-believes E and everyone p-believes that everyone p-believes E and so on ad infinitum. Hence the set of states where E is common p-belief (denoted by $C^p E$) is given as:

$$C^p E = \cap_{n \geq 1} (B_1^p E \cup B_2^p E)$$

2.3.3 The Effect of Beliefs in a Global Game

With the apparatus just defined we can give a precise demonstration of the effect of beliefs on equilibria. We define an event that the signal of player j is higher then t as T.

$$T = \{(\theta, x_1, x_2) : x_j \geq t\}$$

We find how the event $B_i^p T$, the event that player i p-believes that the signal of player j is higher then t, look like. The posterior of player i about the type of player j is $\mathcal{N}(x_i, 2\sigma)$, so player two p-believes T if:

$$1 - \Phi\left(\frac{t - x_i}{\sqrt{2\sigma}}\right) = \Pr(x_j \geq t | x_i) \geq p$$

After solving for x_i we arrive at the condition on player i p-believing event E and get that the event $B_i^p E$ is given as:

$$B_i^p T = \{(\theta, x_1, x_2) : x_i \geq t - \sqrt{2\sigma}\Phi^{-1}(1-p)\} \tag{2.5}$$

We will use this result to see the effect of higher order beliefs (i.e. beliefs about beliefs) about the actions. Revolting is dominant if the parameter θ is higher than γ. We investigate the beliefs and higher order beliefs about an event that the revolting is dominant. We will denote the event as D. If players p-belief the event they know that revolt would not benefit him with probability at least p. The event D can be written as:

$$D = \{(\theta, x_1, x_2) : \theta \geq \gamma\}$$

'The player j p-believes the event D if:

$$1 - \Phi\left(\frac{\gamma - x_j}{\sqrt{\sigma}}\right) = \Pr(\theta \geq \gamma | x_j) \geq p$$

We solve for x_j and get the event that player j p-believes that abstaining is dominated at:

$$B_i^p D = \{(\theta, x_1, x_2) : x_j \geq \gamma - \sqrt{\sigma}\Phi^{-1}(1-p)\} \tag{2.6}$$

We can progress to beliefs about beliefs, the event that a player p-believes that other one p-believes D. By putting together 2.5 and 2.6 we get the set of states $B_1^p B_2^p D$ which characterize the second order beliefs that player one p-believes that the player two p-believes D:

$$B_1^p B_2^p D = \{(\theta, x_1, x_2) : x_1 \geq \gamma - (\sqrt{\sigma} + \sqrt{2\sigma})\Phi^{-1}(1-p)\}$$

And similarly we can continue further. We define the event that player i $2n$th order p-beliefs D recursively as:

$$(B_i^p B_j^p)^n D = B_1^p B_2^p (B_1^p B_2^p)^{n-1} D, \quad i \neq j$$

in our case it is given as:

$$(B_1^p B_2^p)^n D = \{(\theta, x_1, x_2) : x_1 \geq \gamma - \left(\sqrt{\sigma} + (2n-1)\sqrt{2\sigma}\right)\Phi^{-1}(1-p)\} \quad (2.7)$$

As the n grows the event $(B_1^p B_2^p)^n D$ either grows or contracts depending on the value of p. For p smaller then one half the event grows. Thus with small probability p, at any state there will always be some n that a player will $2n$th order p-believe the event D. No matter how small the variance of the signal is, the p-belief of some order that the revolting dominates will spread to all states. For p greater then one half the event contracts. The fact that $(B_1^p B_2^p)^n D$ contracts for high p means that any event like D will be nowhere common p-belief.

Morris *et al.* (1995) discussed how such expansion of beliefs leads to selection of unique equilibrium when an equilibrium is risk dominant at all states. We apply their arguments to our linear example. We will denote the region of states where player i will revolt given that player j revolts with probability p as R_i^p. Hence we have that:

$$R_i^p = \{(\theta, x_1, x_2) : x_i \geq \gamma(1-p)\}$$

Take some $p < \frac{1}{2}$. At states $B_1^p D$ player one believes that with probability at least p he will profit from revolt and at R_1^p this probability p of success is enough for him to revolt. Thus he will revolt at $B_1^p D \cap R_1^p$. So as long as

$$\gamma(1-p) < \gamma - \sqrt{\sigma}\Phi^{-1}(1-p)$$

we have that $B_1^p D \cap D_1^p = B_1^p D$. So he finds it dominant to revolt at $B_1^p D$.

At states $B_2^p B_1^p D$ player two believes with probability at least p that $B_1^p D$ hence at $B_2^p B_1^p D$ player two p-beliefs that player one revolts. As long as

$$\gamma(1-p) < \gamma - (\sqrt{\sigma} + \sqrt{2\sigma})\Phi^{-1}(1-p)$$

$B_1^p D \cap R_2^p = B_2^p B_1^p D$ and hence player two will revolt at $B_2^p B_1^p D$. By continuing this process we iteratively remove the dominated strategies and the region where it is dominated to revolt is getting larger and larger as long as after n iterations following condition holds:.

$$\gamma(1-p) < \gamma - \left(\sqrt{\sigma} + (n-1)\sqrt{2\sigma}\right)\Phi^{-1}(1-p) \tag{2.8}$$

For $p < \frac{1}{2}$ the right hand side of the inequality 2.8 decreases to $-\infty$. With the increasing n for p that are close to one half from below the right hand decreases to the value $\gamma(1-p)$ and hence spread of the beliefs eliminates abstaining for all types in R_i^p. Note that when revolting is risk dominant equilibrium at some state, the state is in R_i^p for some $p < \frac{1}{2}$. So the spread of the beliefs will eliminate abstaining from all states where revolting is risk dominant. The result of global game that players that players select risk dominant strategies is a consequence of the lack of high common belief that playing both strategies is not dominated, so weak (risk dominated) equilibria cannot survive.

2.4 Summary

In this chapter we identified the coordination problem inherent in the collective collection. In the collective action situation there are multiple self-fulling equilibria and it is hard to predict what actions would players select. Often the players face trade off between actions that appear to be safe and those that are Pareto efficient.

When very small idiosyncratic noise is added to the game the safe equilibrium is always selected. The lack of common knowledge about the payoffs causes uncertainty that eliminates the unsafe equilibria. This suggest that commonly shared informations will have large impact on the outcome.

In the next chapter we will consider models of coordination in large population. We will consider several sources of uncertainty. As the private signals need not be always very precise, we will consider the behaviour under uncertainty also away from the limit. This will bring in the role for other sources of information that could be acquired by publicly observed signals.

Chapter 3

The Model of Collective Action

In this chapter we investigate model of coordination that is a continuum player counterpart of the coordination game from previous chapter. Similarly to the previous chapter we describe the model explicitly as an attempt to overthrow government although the setting is far more general and in chapter 5 we will discuss other possible application. In the model we assume that only a certain proportion of the players must revolt to overthrow the government.

In the two subsequent chapters we examine what is the effect of introducing uncertainty into the model. The introduction of small uncertainty in the 2×2 coordination game led to a selection of an unique equilibrium. One might wonder whether introducing uncertainty in the same fashion as in the global game in section 2.2 will also lead to the unique prediction.

We will look at two different sources of uncertainty, one about the strength of the current regime and the other one that stems from heterogeneity of payoffs. We compare the effects of these uncertainties under different information structures.

3.1 Setup of the Model

We assume there is a continuum of players of measure one, who try to overthrow a government. They choose between two actions: revolting or abstaining. The the players must collectively revolt in order to achieve the goal of overthrowing the government. Enough people must revolt for the attempted revolution to be successful. More specifically, the regime is overthrown if the proportion of players who revolt (denoted by M) is bigger than a threshold denoted by T.

We assume that the payoffs of the agents are the same as in the example

	$M \geq T$	$M < T$
revolt	B	$-c$
abstain	0	0

Table 3.1: The Payoffs in the Model of Collective Action

from previous chapter. A player receives some private benefit of the utility of B if he participates on a successful revolution. If he participates on an unsuccessful attempt to overthrow the government, he is punished and his welfare reduces to $-c$. We summarize the payoffs in the table 3.1. Moreover in chapter 5 we discuss the model in the setting of the revolution as well as in the context of other application.

3.1.1 Signals and Beliefs

We will investigate the game of incomplete information, where the parameters of the game depend on a random variable θ. By the principle of insufficient reason we suppose that players have common improper uniform prior about θ. Therefore the prior density of θ is constant. Although such density is not a probability distribution, the posterior will be well-defined probability distribution. The fact that prior is not probability distribution does not cause any problems.

We assume that the get noisy signals about the realization of θ. The signals come in two forms. There are idiosyncratic signals that differ across the players and there is common signal that is shared by all players. Moreover we assume that the common signals is not only observed by all, but its realization is also common knowledge.

More specifically we assume that noise in the signals is normally distributed. Each player receives a private signal denoted as x. We assume that conditionally on the realization of θ the signals in the population are distributed normally with mean θ and variance $1/\sigma$. In other words randomly selected person's private signal has distribution $\mathcal{N}(\theta, 1/\sigma)$.

The reason why we use the variance in the form of a fraction is to make normal Bayesian updating easier. We define precision as inverse of variance. Hence the normal distribution of the signals has precision σ. Based just on his private signal the posterior about θ of a player with a signal x is $\mathcal{N}(x, 1/\sigma)$.

In addition we look at the effect of the public signal. The public signal y is a random variable and $y = \theta + \epsilon$, where epsilon is the noise term distributed according $\mathcal{N}(0, 1/\eta)$. When the players receive only the common signal y, their posterior about θ is $\mathcal{N}(y, 1/\eta)$.

When both signals are available, the posterior is given by standard normal Bayesian updating. One can find the posterior by using the Bayes' rule for density functions. We describe the detailed derivation in the Appendix. After receiving public signal y and private signal x, the posterior about θ is normal and given by:

$$\mathcal{N}\left(\frac{\eta}{\eta+\sigma}y + \frac{\sigma}{\eta+\sigma}x, \frac{1}{\eta+\sigma}\right)$$

We will fix the value of the public signal and treat it as a parameter. To avoid writing the lengthy notation for the posterior we will sometimes denote its cumulative density function given signal x and y as $Q(\theta|x)$. Note that Q is increasing in x in the sense of first order stochastic dominance. In other words $x > x'$ implies that $Q(\theta|x') < Q(\theta|x)$.

3.1.2 Strategies and Equilibrium

The type of each player is the private signal x that he received. A strategy assigns an action to each type of a player. In what follows we will consider only pure and symmetric strategies. To make the analysis even simpler we also assume that indifferent types choose to revolt. We can characterize the strategy as a set S of types who revolt. Hence the size of the action given a strategy S is:

$$M(S|\theta) = \int_S \sqrt{\sigma}\phi\big(\sqrt{\sigma}(x-\theta)\big)dx \tag{3.1}$$

Without the private signal everyone is of the same type, hence there are only two strategies possible, everyone revolts or everyone abstains and accordingly the size of action is either zero or one. The expected utility from revolting is given the types in S revolt is:

$$U(x, S) = \mathbb{E}\Big(\mathbb{I}_{M(S|\theta)\geq T}B - \mathbb{I}_{M(S|\theta)<T}c \;\Big|\; x\Big)$$

We can define a best response and an equilibrium.

Definition 3.1. Strategy $BR(S)$ is a best response to S if:

$$BR(S) = \{x \in \mathbb{R} : U(x, S) \geq 0\}$$

Definition 3.2. Strategy S is an equilibrium if $BR(S) = S$.

3.2 Complete Information

First we look at the benchmark case with complete information. We suppose that all the parameters of the game are common knowledge. There are several possible equilibria in dependence on the parameters. If the payoff B is negative, nobody would benefit from participating in the revolution, so the revolting is dominated and the only Nash equilibrium is the one, in which everyone abstains.

When the payoff B is positive, the equilibria depend on the parameter T. If $T > 1$, then independently of the mass of the people revolting the revolution will never be successful, hence revolting is dominated. and there is unique Nash equilibrium, in which everyone abstains. In the last trivial case when $T \leq 0$, the revolution is successful even if nobody participates in it. Then the action abstain is dominated and in the unique equilibrium strategy everyone revolts.

The more interesting case when $0 \leq B$ and $0 < T \leq 1$ gives rise to multiple equilibria. When players use only pure strategies, the probability that the regime is overthrown is either zero or one. If it is zero, the best response is for everyone to abstain, which further leads to the zero chance of a successful revolt. If the probability of regime change is one, everyone revolts and the regime collapses with certainty. Hence there are two self-fulfilling pure strategy equilibria. In the nontrivial case the outcome is indeterminate and depends only on the expectations of the players.

3.3 Uncertainty about the Threshold

In this section we look at the situation when there is an uncertainty about the strength of the government. In this case the players are unsure about the exact value of the threshold T. The value of the random parameter determines the value of the threshold as $T = \theta$. All other parameters B and c are common knowledge. The players observe signals about the parameter θ that we described in section 3.1.1. We will look at three cases: one in which players receive both the idiosyncratic and the common signals and other two extreme cases with only the common or only the idiosyncratic signals. Similarly to baseline global game, the information structure with the individual signals generates uncertainty not only about parameter T but also about other players' informations and this uncertainty about higher order beliefs narrows the predictions of the game.

This model is one of a classic example of global game with continuum of players. We follow the exposition of the case with private and public information that can be found in many sources (Edmond 2011; Morris & Shin 2004; 2001; 2000). We further discuss the case with only the public information.

3.3.1 Solution

In this section we will search for the equilibria with private signals. We will search for an equilibrium in specific form characterized by two cutoff levels $\tilde{x}$ and $\tilde{T}$, that are such that the equilibrium strategy is to revolt if the type x of a player is weakly smaller than the cutoff point $\tilde{x}$ and given this strategy the regime collapses if the threshold T is weakly smaller than the cutoff point $\tilde{T}$. It turns out that there is indeed such equilibrium. In addition in many cases this strategy is the only one surviving the iterated elimination of strictly dominated strategies. To verify that the pair is equilibrium we first suppose the regime falls if $T \leq \tilde{T}$. Then the expected utility of revolting of type x is:

$$\begin{aligned}&\Pr(T \geq \tilde{T} \mid x)B - \big(1 - \Pr(T \geq \tilde{T} \mid x)\big)c = \\ &= \Pr(T \geq \tilde{T} \mid x)(B + c) - c\end{aligned}$$

After we plug in the posterior the expected utility from revolting becomes:

$$Q(\tilde{T}|x)(B + c) - c$$

Each type will revolt if the expected payoff from revolting will be weakly greater than zero. Because the posterior is strictly increasing (in the sense of first order stochastic dominance) in the private signal x, the payoff from revolting is decreasing in x and therefore only small types will revolt. Hence given the regime falls if $T \leq \tilde{T}$, only the types weakly smaller than $\tilde{x}$ will participate in the revolution and the cutoff point $\tilde{x}$ satisfies the indifference condition:

$$Q(\tilde{T}|\tilde{x})(B + c) = c \tag{3.2}$$

Next we show that given the monotone strategy $(-\infty, \tilde{x}\rangle$, the regime falls whenever $T \geq \tilde{T}$. Given the threshold T the size of the action of the cutoff strategy is:

$$M((-\infty,\tilde{x}\rangle \mid T) = \int_{-\infty}^{\tilde{x}} \sqrt{\sigma}\phi\big(\sqrt{\sigma}(x-T)\big)dx = \\ = \Phi\big(\sqrt{\sigma}(x-T)\big)$$

and the regime collapses if

$$\begin{aligned} M((-\infty,\tilde{x}\rangle \mid T) &\geq T \\ \Phi\big(\sqrt{\sigma}(\tilde{x}-T)\big) &\geq T \\ \Phi\big(\sqrt{\sigma}(\tilde{x}-T)\big) - T &\geq 0 \end{aligned}$$

Since the left hand side of the inequality is decreasing, the regime falls for T weakly smaller than $\tilde{T}$ and the cutoff level $\tilde{T}$ solves:

$$\Phi\big(\sqrt{\sigma}(\tilde{x}-\tilde{T})\big) = \tilde{T} \tag{3.3}$$

Thus in equilibrium the pair of cutoffs $\tilde{x}$ and $\tilde{T}$ must solve the conditions 3.3 and 3.2. To solve the two equation we find $\tilde{x}(\tilde{T})$ that solves 3.2 for each $\tilde{T}$ and then substitute $\tilde{x}(\tilde{T})$ for $\tilde{x}$ into the equation 3.3.

3.3.2 Solution with Private Information Only

The absence of the public signal considerably simplifies the analysis. Without the public signal y the posterior distribution of T is normal with mean x and precision σ. Hence:

$$Q(\tilde{T}|\tilde{x}) = \Phi\big(\sqrt{\sigma}(\tilde{T}-\tilde{x})\big)$$

Therefore the condition for the marginal type 3.2 given $\tilde{T}$ is:

$$\Phi\big(\sqrt{\sigma}(\tilde{T}-\tilde{x})\big)(B+c) = c$$

After some algebraic manipulation we solve for $\tilde{x}$ and we can characterize the function $\tilde{x}(\tilde{T})$ as:

$$\tilde{x}(\tilde{T}) = \tilde{T} - \frac{1}{\sqrt{\sigma}}\Phi^{-1}\left(\frac{c}{B+c}\right)$$

And plugging it into the equilibrium equation 3.2 we get:

$$\begin{aligned}
\Phi\left(\sqrt{\sigma}\left(\tilde{T} - \frac{1}{\sqrt{\sigma}}\Phi^{-1}\left(\frac{c}{B+c}\right) - \tilde{T}\right)\right) &= \tilde{T} \\
\Phi\left(-\Phi^{-1}\left(\frac{c}{B+c}\right)\right) &= \tilde{T} \\
1 - \frac{c}{B+c} &= \tilde{T} \\
\frac{B}{B+c} &= \tilde{T}
\end{aligned}$$

The last equality characterizes the equilibrium cutoff level $\tilde{T}$. Thus in the equilibrium whenever the realization of T will be above $\frac{B}{B+c}$ the regime will collapse.

3.3.3 Solution with Private and Public Information

The availability of the public signal slightly complicates the analysis. The cumulative distribution function $Q(\tilde{T}|\tilde{x})$ of the posterior is given as:

$$\Phi\left(\sqrt{\eta+\sigma}\left(\tilde{T} - \frac{\eta}{\eta+\sigma}y - \frac{\sigma}{\eta+\sigma}\tilde{x}\right)\right)$$

by plugging it into the indifference condition 3.2 we get:

$$\Phi\left(\sqrt{\eta+\sigma}\left(\tilde{T} - \frac{\eta}{\eta+\sigma}y - \frac{\sigma}{\eta+\sigma}\tilde{x}\right)\right)(B+c) = c$$

and solving for $\tilde{x}$ we get the formula for the function $\tilde{x}(\tilde{T})$:

$$\tilde{x}(\tilde{T}) = \frac{\eta+\sigma}{\sigma}\tilde{T} - \frac{\eta}{\sigma}y - \frac{\sqrt{\eta+\sigma}}{\sigma}\Phi^{-1}\left(\frac{c}{B+c}\right) \tag{3.4}$$

To get the $\tilde{T}$ that arises in equilibrium, we plug 3.4 into the equilibrium condition 3.3. Thus in the equilibrium the cutoff level $\tilde{T}$ must solve:

$$\Phi\left(\sqrt{\sigma}\left(\frac{\eta+\sigma}{\sigma}\tilde{T} - \frac{\eta}{\sigma}y - \frac{\sqrt{\eta+\sigma}}{\sigma}\Phi^{-1}\left(\frac{c}{B+c}\right) - \tilde{T}\right)\right) = \tilde{T}$$

And after subtracting terms with $\tilde{T}$ inside the brackets the equilibrium condition reduces to:

$$\Phi\left(\sqrt{\sigma}\left(\frac{\eta}{\sigma}(\tilde{T}-y)-\frac{\sqrt{\eta+\sigma}}{\sigma}\Phi^{-1}\left(\frac{c}{B+c}\right)\right)\right)=\tilde{T} \tag{3.5}$$

Contrary to the simple case without the shared signal we cannot give the exact value that solves the equation 3.5, but at least we can say that there is some solution. Since the left hand side of the equation is bounded and the right hand side is unbounded and both of the sides are continuous, they must cross at least once. Moreover we can give a sufficient conditions for the uniqueness of the solution.

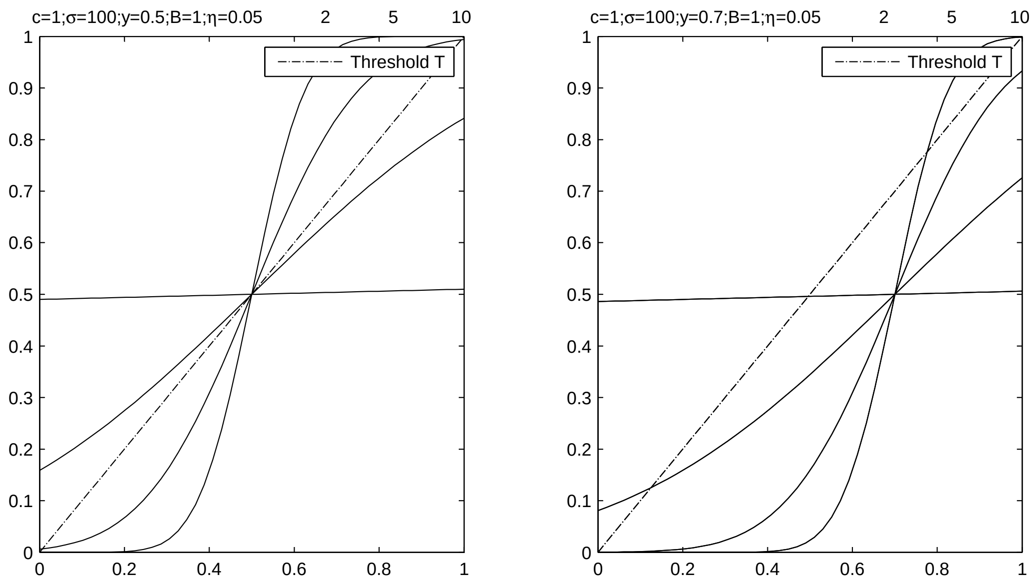

Figure 3.1: The Effect of the Precision of the Public Signal

Claim 3.1. *If $2\pi\eta < \sqrt{\sigma}$, there is only one value of $\tilde{T}$ that solves the equation 3.5.*

Proof. If the difference of the LHS and RHS of the equation 3.5 is monotonic in $\tilde{T}$, then the solution is unique. We differentiate the difference with respect to $\tilde{T}$ and find under what conditions it is monotonic. The difference is strictly decreasing if and only if:

$$\frac{d}{dx}\Phi\left(\sqrt{\sigma}\left(\frac{\eta}{\sigma}(\tilde{T}-y)-\frac{\sqrt{\eta+\sigma}}{\sigma}\Phi^{-1}\left(\frac{c}{B+c}\right)\right)\right)-\tilde{T}<0$$

$$\frac{\eta}{\sigma}\sqrt{\sigma}\phi\left(\sqrt{\sigma}\left(\frac{\eta}{\sigma}(\tilde{T}-y)-\frac{\sqrt{\eta+\sigma}}{\sigma}\Phi^{-1}\left(\frac{c}{B+c}\right)\right)\right)-1<0 \tag{3.6}$$

Since $\max\phi(\cdot)=2\pi$, 3.6 is satisfied if:

$$\frac{\eta}{\sqrt{\sigma}}2\pi-1<0$$
$$2\pi\eta<\sqrt{\sigma}$$

□

The fact that there is only one cutoff equilibrium also means that it is the only equilibrium of the game. This is obtained by the similar process as in the subsection 2.2.2. Because the best response function is increasing, the iterated elimination of dominated strategies will narrow the scope of possible equilibria and find the lowest and highest equilibrium. When the private information is sufficiently precise the highest and lowest equilibria coincide.

Claim 3.2. *If $2\pi\eta<\sqrt{\sigma}$, then the game has unique equilibrium.*

Proof. We show the uniqueness by iterated elimination of strictly dominated strategies. Suppose there is a strategy such there is cutoff point t that the regime collapses for all $T\leq t$. Then the best response to the strategy is the cutoff strategy $S=(-\infty,\tilde{x}(t)\rangle$. When agents play the strategy, then for the regime will collapse for all T smaller than t' that solves:

$$\Phi\big(\sqrt{\sigma}(\tilde{x}(t)-t')\big)=t' \tag{3.7}$$

We define an implicit function $b(t)$ as the value t' that solves 3.7 for each t. Since the LHS of 3.7 is decreasing in t, $b(t)$ is increasing in t. The lowest possible value of the cuttoff point t is zero. All types lower than $\tilde{x}(0)$ will have dominant strategy to participate, which in turn means that for all T smaller than $b(0)$ the regime collapses under any undominated strategy. Hence all types smaller than $\tilde{x}\big(b(0)\big)$ find it dominant to revolt, when they play against undominated strategy and the regime will collapse for all T smaller than $b^2(0)$. By iteratively applying the function $b^n(0)$ we get the upper bound on all thresholds T, for which the regime must fall under any undominated strategy. Similarly by iteratively finding $b^n(1)$ we get the lower bound on thresholds, in which the regime will not be overthrown under any undominated strategy.

For the uniqueness we must show that $b^n(1)$ and $b^n(0)$ have the same limit. Because $b(t)$ is continuous the limit of $b^n(0)$ and $b^n(1)$ must be the fixed point of $b(t)$ and we showed that the fixed point is unique the assumption.

□

The uniqueness of the equilibrium depends on the relative precisions of the private and public information. If the public signal is too precise, some common knowledge is restored and the multiplicity game arises as in the complete information game. The figure 3.1 shows the effect of variance on number of solutions to equation 3.5. The dashed line plots the right hand side and the solid line plots the left hand side of the equilibrium condition 3.5. The figure shows the effect of different values of the precision for two values of the public signal ($y = 0.5$ and $y = 0.7$). As the precision of the public signal rises the left hand side of the equation 3.5 becomes steeper and the solid line crosses the dashed line several times. On the other hand as the precision of private information rises, the effect of public signal vanishes, the solid line becomes flatter and the equilibrium $\tilde{T}$ approaches the solution with only the private information.

3.3.4 Effect of the Public Signal

We can look at the effect of the public signal on the equilibrium threshold $\tilde{T}$. The figure 3.2 illustrates the effect of the public signal. The solid lines plots the left hand side of the equilibrium condition 3.5 for various values of the public signal y. As the public signal grows, the posterior distribution rises in the sense of first order stochastic dominance, the left hand side of the equation 3.5 shifts rightwards, and equilibrium cutoff threshold $\tilde{T}$ decreases. Moreover if the public signal is precise, then for intermediate values of y multiple equilibria appear. The public information influences the outcome, although it does not affect the fundamental parameter of the game.

3.3.5 Solution with Public Information Only

In the end we investigate the case, when the players receive only the public signal. Since the signal is common knowledge, all types are identical and in a symmetric equilibrium everyone takes the same action. As in the complete information game the size of the action is either one or zero. First we will suppose that everyone revolts and the size of action is one. The strategy will

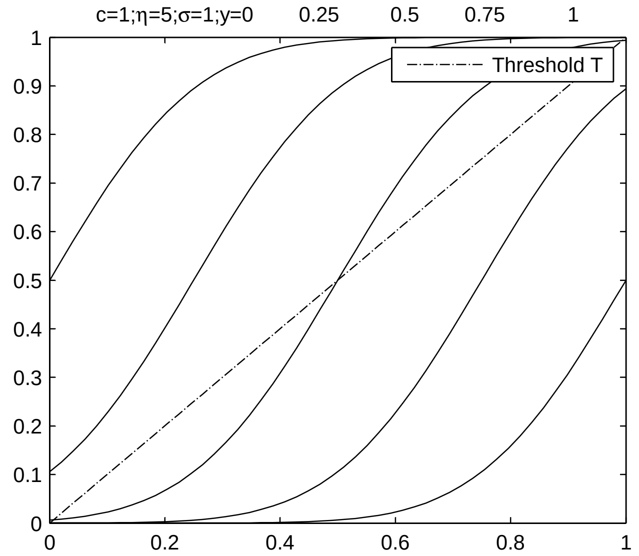

Figure 3.2: The Effect of the Public Signal

result in a regime change as long as $T \leq 1$. So given the signal y the revolution will be successful with probability:

$$\Phi\big(\sqrt{\eta}(1-y)\big)$$

Thus the players will expect to benefit from the revolt if:

$$\Phi\big(\sqrt{\eta}(1-y)\big)(B+c) \geq c$$

Since the left hand side of the inequality is decreasing, the inequality will be satisfied for y weakly smaller than $\overline{y}$, where $\overline{y}$ solves the indifference condition:

$$\Phi\big(\sqrt{\eta}(1-\overline{y})\big)(B+c) = c$$

Similarly we find the lowest signal for which the players will abstain. Suppose that all players abstain. Then the status quo changes if $T \leq 0$. So the probability of change is:

$$\Phi\big(\sqrt{\eta}(0-y)\big)$$

Nobody will have an incentive to deviate and revolt as long as the expected benefit from revolting is smaller than zero, so if the following holds:

$$\Phi\big(\sqrt{\eta}(0-y)\big)(B+c) < c$$

The inequality will be satisfied for all y that are higher than $\underline{y}$ that solves:

$$\Phi\big(\sqrt{\eta}(-\underline{y})\big)(B+c) = c$$

There are multiple equilibria if $y \in (\underline{y}, \overline{y})$. As the noise of the public signal vanishes, there will be both equilibria if the value of the signal is between one and zero. For signals lower than zero revolting will be dominant and for signals higher than one abstaining will be dominant. Because the value of public signal is common knowledge, contrary to the private signal, the increased precision of public signal will lead to the same outcomes as in the complete information game.

3.4 Summary

The results for the model with only private signals parallel the qualitative results of the 2×2 global game in the section 2.2. The introduction of idiosyncratic uncertainty lead to uniqueness equilibria. Moreover the unique equilibrium can be found by iterated elimination of strictly dominated strategies. There are three key properties that are common to the linear example in the section 2.2 and to the model with continuum population. The payoffs from revolting depend monotonically on the state, the payoffs are increasing in the mass of types who revolt and both actions revolting and abstaining are dominant for very high or very low states.

The first two traits ensure that best response to a cutoff strategy will be again a cutoff strategy. The second point means that both of the two games are of strategic complements. Because of the third point there is a region, where the iterated deletion of the dominated strategies can start. Summing these traits together we can find lowest and highest equilibria in both games by the iterated elimination of strictly dominated strategies.

These properties, however, do not ensure that the equilibrium is unique. Other point that drives the uniqueness is the precision of the idiosyncratic signals. As the noise of the private signal vanishes, the private signal become very informative. The fact that the signal are received idiosyncratically will, however, generate large higher order uncertainty about beliefs about beliefs. The large higher order uncertainty in combination with almost certainty about the states causes that the process of iterated deletion of dominated strategies eliminates all but one strategy. Morris & Shin (2001) further show that the results about uniqueness hold quite generally. As long as the three properties are satisfied than for a general prior as the noise of the private signal vanishes there is a unique cutoff strategy.

The opposite situation happens when the noise gets smaller in the public signal. Consider the case with only the public signal in section 3.3.5. All players are absolutely sure about the state and also about the beliefs of other players, so as the noise in the common signal disappear all players will be sure about the higher order beliefs and almost certain about the state. When the value of the public signal is between zero and one it will be almost common knowledge that both revolting and abstaining are undominated actions. Thus the multiplicity of equilibria arises as in the complete information game.

In the case with both signals the public signal influences the decisions and low public signal induces more types to revolt. The introduction of the relatively precise common signal, however, leads again to multiplicity of equilibria. As the public signal becomes relatively informative, the agents will base their decisions on an information that is common knowledge and the role of higher order beliefs diminishes.

Chapter 4

Heterogeneity in Payoffs

In this chapter we introduce heterogeneity in the payoffs into the model of previous chapter. We look at a stripped down variant of a model of Mesquita (2010). In this variant of the model the threshold T and the cost of unsuccessful revolution c are common knowledge. The uncertainty in this model stems from the heterogeneity of the payoffs. The payoffs are distributed in the society according a normal function. Each player knows his payoff, but he does not know the exact distribution of the payoffs in the society. His payoff, however, serves as a signal about the distribution.

It turns out that the key difference between the uncertainty about the payoffs and the uncertainty about the threshold is that in the former case there are no states, where it would be dominant to revolt. There is what Mesquita (2010) calls one-sided dominance as opposed to two-sided dominance in the previous version of the model. Thus the success of a revolution always depends on the participation of others. As a consequence we loose on one hand the convenience of having unique equilibrium, on the other hand there will always be an equilibrium with no type participating in the revolution. The multiplicity of equilibria can thus explain sudden change from an absolutely calm situation to a situation, when people actively revolt.

4.1 The Setup

We will take the setting of the model from last chapter, but we assume the payoffs B are heterogeneous. The distribution of the payoffs in the population depend on the random variable θ. Conditional on the random variable θ the payoffs are distributed according to a normal distribution $\mathcal{N}(\theta, 1/\sigma)$.

	$M \geq T$	$M < T$
revolt	x	$-c$
abstain	0	0

Table 4.1: The Payoffs of a Player of Type x

The parameters c and T are common knowledge. The shape of the distribution of the payoffs is commonly known, but the mean of the distribution θ is uncertain. Each player knows his own payoff, but he does not know the exact realization of θ and thus the exact distribution of the payoffs in the population.

The payoff of a player is also his type. The type of a player, informs the player about two things. Firstly the type of a player gives him the value of his payoff. Secondly the type of a player serves as a signal about the realization of the parameter θ. Hence the private signal of a player is also his the payoff i.e. $B = x$. We summarize the payoffs in the table 4.1.

In addition to the private signal, we assume that players receive a public signal y that is commonly observed and does not affect any of the parameters of the game. We assume that the signal comes in the form $y = \theta + \epsilon$, where the random noise ϵ is distributed according $\mathcal{N}(0, \eta)$. So the signal structure is the same as in the subsection 3.1.1. Thus the posterior about θ is:

$$\mathcal{N}\left(\frac{\eta}{\eta+\sigma}y + \frac{\sigma}{\eta+\sigma}x, \frac{1}{\eta+\sigma}\right)$$

To make the notation shorter we will sometimes label the posterior cumulative distribution function of θ as $Q(\theta|x)$. We already noted that Q is increasing in x in the sense of first order stochastic dominance.

Given the strategy S and the associated size of action $M(S \mid \theta)$ the expected utility from revolting is given as:

$$\begin{aligned}
U(x,S) &= \mathbb{E}\Big(\mathbb{I}_{M(S|\theta)\geq T}B - \mathbb{I}_{M(S|\theta)<T}c \;\Big|\; x\Big) = \\
&= \Pr\big(M(S \mid \theta) \geq T \mid x\big)x - \Pr\big(M(S \mid \theta) < T \mid x\big)c = \\
&= \Pr\big(M(S \mid \theta) \geq T\big)(x+c) - c
\end{aligned}$$

4.2 Complete Information

In this subsection we consider the benchmark case of complete information. We suppose that parameter θ is common knowledge. First we consider the strategy that no type participates in the revolution. Then the size of the action is zero and the expected payoff conditional on others playing the strategy is:

$$U(x, \emptyset) = 0 \times (x + c) - c = -c < 0$$

Hence the best response is to never revolt and therefore the strategy constitutes an equilibrium. Thus for any of value of θ there is always an equilibrium, in which no type revolts.

In addition there could be another equilibrium where some types participate. Since all players play only pure strategies and the game is deterministic the outcome of the game is also deterministic, so the probability that the regime falls is either one or zero. We already considered the latter case. We suppose that the strategy S is such that the regime changes with probability one. Then expected payoff is:

$$U(x, S) = 1 \times (x + c) - c = x$$

Hence everyone who has positive payoff x will participate. So, as long as

$$M(\langle 0, \infty) \mid \theta) = 1 - \Phi\big(\sqrt{\sigma}(0 - \theta)\big) \geq T \tag{4.1}$$

holds, the regime colapses with probability one and there is another equilibrium with positive participation. Because the right hand side of the inequality 4.1 is increasing in θ, there is an equilibrium with positive participation only for high values of θ.

4.3 Incomplete Information

In this section we look at model with uncertainty about the parameter θ that we outlined at the beginning of the chapter. We consider the cases, when the only source of the players' information is their type (the private information) as well as when they have additional public signal.

There is always an equilibrium where nobody participates. The argument is the same as for the complete information case. If nobody participates, the probability of change is zero. Thus, the expected profit from revolting is negative and nobody wants to participate. This gives rise to self fulling equilibrium. In this respect our game differs from the continuum player global game of section 3.3.2 and because of this feature there are multiple equilibria.

In the rest of the section we characterize other equilibria. We will again search for equilibria in a special form. We look for an equilibrium characterized by a pair cutoff points $\tilde{x}$ a $\tilde{\theta}$, that are such that in an equilibrium a player participates if his type is higher than $\tilde{x}$ and the regime collapses if the parameter θ is higher than $\tilde{\theta}$. We will call such strategies cutoff or monotone strategies. Below, we verify that such strategy constitutes equilibrium and characterize the cutoff $\tilde{\theta}$.

We follow the derivation of Mesquita (2010), but we solve for $\tilde{\theta}$ rather than for $\tilde{x}$. This method is slightly more complicated, but it provides more direct comparison with the model of previous chapter and allows us to easily investigate the best response dynamics. In addition to the model of Mesquita (2010) we discuss the case without the public signal and elaborate on the best response dynamics.

We fix the signal y and suppose there is such strategy $\tilde{S}$ that status quo falls if $\theta \geq \tilde{\theta}$. The player's expected utility from participation is given by:

$$
\begin{aligned}
U(x, \tilde{S}) &= \Pr(\theta > \tilde{\theta} \mid x)(x + c) - c = \\
&= \big(1 - Q(\tilde{\theta} \mid x)\big)(x + c) - c
\end{aligned}
\tag{4.2}
$$

The type x will participate if his expected utility from participation will be greater than zero. If $x \leq 0$ the expression in 4.2 is negative, hence no types smaller than 0 will participate. For $x > 0$ the utility of revolting is increasing in x, so the player will participate if his type is greater than $\tilde{x}$, where the cutoff point $\tilde{x}$ solves the indifference condition:

$$
\big(1 - Q(\tilde{\theta} \mid \tilde{x})\big)(\tilde{x} + c) = c \tag{4.3}
$$

Given the monotone strategy $\langle \tilde{x}, \infty)$ the size of the action at state θ is:

$$
\begin{aligned}
M(\langle \tilde{x}, \infty) \mid \theta) &= \int_{\tilde{x}}^{\infty} \sqrt{\sigma}\phi\big(\sqrt{\sigma}(x - \theta)\big)dx = \\
&= 1 - \Phi\big(\sqrt{\sigma}(\tilde{x} - \theta)\big)
\end{aligned}
$$

The regime will fall if the size of action is greater than the threshold T. The size of action is increasing in the parameter θ. Hence given the cutoff strategy the regime will fall if the parameter θ is larger than $\tilde{\theta}$ that solves the indifference condition:

$$1 - \Phi\big(\sqrt{\sigma}(\tilde{x} - \tilde{\theta})\big) = T \tag{4.4}$$

Thus in the equilibrium the pair of cutoff points $\tilde{x}$ and $\tilde{\theta}$ must simultaneously solve the equations 4.3 and 4.4. First we solve it for the easier case with only private information.

4.3.1 Solution with Private Information Only

The solution with only the idiosyncratic signals is easier. We plug in for the posterior in 4.3 and we get that $\tilde{x}$ and $\tilde{\theta}$ must solve these two equations:

$$1 - \Phi\big(\sqrt{\sigma}(\tilde{x} - \tilde{\theta})\big) = T \tag{4.5}$$

$$\Big(1 - \Phi\big(\sqrt{\sigma}(\tilde{\theta} - \tilde{x})\big)\Big)(\tilde{x} + c) = c \tag{4.6}$$

The equation 4.5 could be alternatively written as:

$$\Phi\big(\sqrt{\sigma}(\tilde{\theta} - \tilde{x})\big) = T \tag{4.7}$$

We plug the equation 4.7 into the equation 4.6 and we get that for the cutoff level $\tilde{x}$ it must hold:

$$(1 - T)(\tilde{x} + c) = c$$

$$\tilde{x} = \frac{cT}{1 - T}$$

We obtain the value of $\tilde{\theta}$ by plugging $\tilde{x}$ back into the equation 4.5:

$$\tilde{\theta} = \frac{cT}{1 - T} - \frac{1}{\sqrt{\sigma}}\Phi^{-1}(1 - T) \tag{4.8}$$

Thus in the case with only idiosyncratic signals we found one cutoff equilibrium. Contrary to the model in previous chapter, however, this equilibrium is not unique, because there is always an equilibrium, in which no types participate.

4.3.2 Solution with Private and Public Information

When there are public signals available, the solution becomes more complicated. We will define the function $\tilde{x}(\tilde{\theta})$. $\tilde{x}(\tilde{\theta})$ is $\tilde{x}$ that implicitly solves 4.3 for a value of $\tilde{\theta}$. We plug it into 4.4 and we get that in equilibrium $\tilde{\theta}$ has to solve:

$$1 - \Phi\big(\sqrt{\sigma}(\tilde{x}(\tilde{\theta}) - \tilde{\theta})\big) = T \tag{4.9}$$

We define the function $A(\theta)$ as:

$$A(\theta) = 1 - \Phi\big(\sqrt{\sigma}(\tilde{x}(\theta) - \theta)\big)$$

We cannot give exact value $\tilde{\theta}$ that solves the equation 4.9, but we can characterize some qualitative behaviour. In an equilibrium in the form of a monotone strategy, the cutoff point $\tilde{\theta}$ must solve $A(\tilde{\theta}) = T$. The figure 4.1 plots the function $A(\theta)$ and the threshold T for the cases with and without the public signal. In the case with public information the function has the shape of peak and there are two cutoff equilibria. In the case with the private information the function is increasing and it crosses th line T at the single private equilibrium that we derived in derived in the previous subsection. This observations holds more generally.

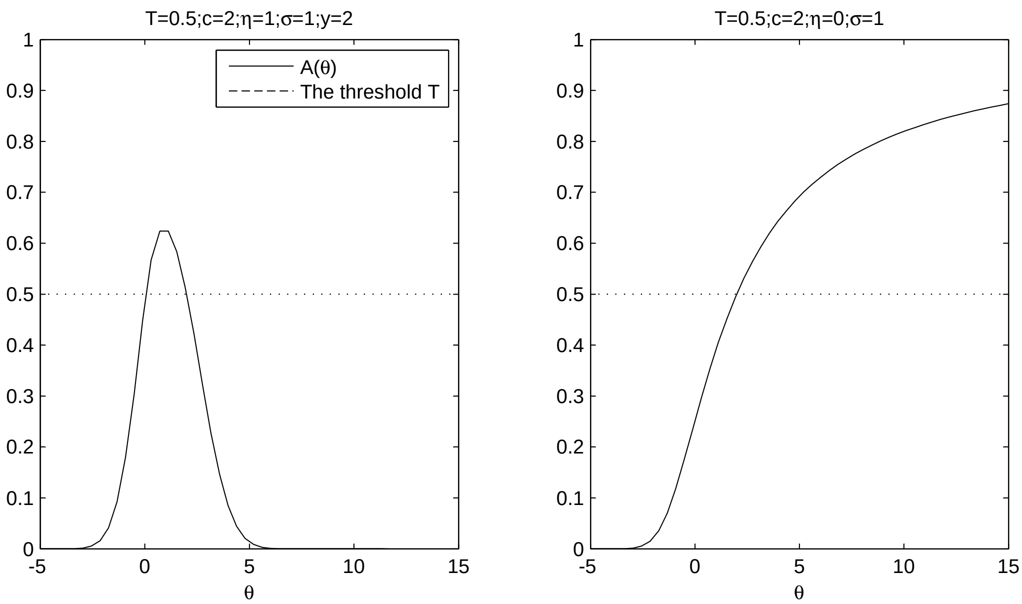

Figure 4.1: Function $A(\theta)$ and the Threshold T

Claim 4.1. *If the players receive only private signals, $A(\theta)$ is increasing. If there is public signal available, $A(\theta)$ is single peaked and* $\lim_{\theta \to -\infty} A(\theta) = \lim_{\theta \to \infty} A(\theta) = 0$

The proof proceeds by differentiating the function $A(\theta)$ and examining its slope and it is relegated to the appendix. With public signal, the function $A(\theta)$ crosses the line T at most twice. For some values of parameters, however, the $A(\theta)$ could never reach the value T, and there could be no cutoff equilibrium. Hence there could be at most two cutoff equilibria. We will call the one with lowest cutoff level $\tilde{\theta}_L$ as lowest equilibrium and the one with higher cutoff level $\tilde{\theta}_M$ as middle equilibrium. The highest equilibrium is the one, in which no type participates (with the cutoff level $\tilde{\theta}_H = \infty$).

Mesquita (2010) suggested that one equilibrium could be eliminated to narrow scope of possible outcomes. Below we elaborate on the argument and discuss it in more detail. We show that in the case with public signal, we can eliminate the middle equilibrium.

4.3.3 Best Response Dynamics

We investigate the stability of the equilibria under the best response (Cournot) dynamics. We assume that the game is played repeatedly and the players choose the best responses to the strategy that was selected in the last stage. Fudenberg & Levine (1998) criticized Cournot dynamics, because it presupposes certain nativity of the players. Each player must believe that nobody else will change his action in the stage. Each agent plays best response to the current strategy rather than using more sophisticated procedure to predict others' actions more accurately. Nonetheless, we will exclude the equilibrium that is unstable under the Cournot dynamics, because the best response dynamics could be a reasonable approximation of more sophisticated learning dynamics.

We will look at the dynamics of the monotone strategies. We will model the dynamics as the development of the cutoff level $\tilde{\theta}$. We already showed that when the types play a strategy such that the regime collapses if θ is greater than t, then the best response is a strategy, in which all types greater $\tilde{x}(t)$ revolt. So if the types follow the strategy, the regime collapses if θ is greater than a cutoff b that solves:

$$1 - \Phi\big(\sqrt{\sigma}(\tilde{x}(t) - b)\big) = T \tag{4.10}$$

Thus after some manipulation we get that the cutoff in the next period depend on the past one through the function $b(t)$:

$$b(t) = \tilde{x}(t) - \frac{1}{\sqrt{\sigma}}\Phi^{-1}(1-T) \tag{4.11}$$

Whether the cutoff increases or decreases depend on whether the function $A(\theta)$ is above or under the threshold T. We notice that:

$$\begin{gathered} A(t) > T \\ A(t) = 1 - \Phi\big(\sqrt{\sigma}(\tilde{x}(t)) - t)\big) > 1 - \Phi\big(\sqrt{\sigma}(\tilde{x}(t) - b(t))\big) = T \\ \tilde{x}(t) - t < \tilde{x}(t) - b(t) \\ b(t) < t \end{gathered}$$

and in addition:

$$\begin{gathered} \tilde{x}(t) > \tilde{x}(b(t)) \\ A\big(b(t)\big) = 1 - \Phi\Big(\sqrt{\sigma}(\tilde{x}(b(t))) - b(t))\Big) > 1 - \Phi\Big(\sqrt{\sigma}(\tilde{x}(t) - b(t))\Big) = T \\ A\big(b(t)\big) > T \\ b\big(b(t)\big) < b(t) \end{gathered}$$

So by induction if for t holds that $A(t) > T$, then $b^n(t)$ is decreasing sequence that is bounded below by $\tilde{\theta}_L$ and since $\tilde{\theta}_L$ is a fixed point of $b(t)$, the dynamic process converge to $\tilde{\theta}_L$. Similarly for a t $A(t) < T$ the sequence $b^n(t)$ is increasing. So for $t < \tilde{\theta}_L$ the sequence of $b^n(t)$ will converge to $\tilde{\theta}_L$. Thus when we perturb the cutoff point away from the lowest equilibrium the learning process will converge back to $\tilde{\theta}_L$. On the figure 4.2 we plot several paths of the dynamics. On the vertical axis there are the starting point of the dynamics and the solid lines plot the paths of the best response dynamics in time. We see that for the starting values lower than $\tilde{\theta}_M$ the paths converge to $\tilde{\theta}_L$. While for starting points higher than $\tilde{\theta}_M$ the paths diverge to infinity.

We exclude the middle equilibria, because it is unstable. Moreover, the middle equilibrium reacts intuitively to the change in parameters. For example consider a change in the parameter T, the strength of the government to withstand a rebellion. As the threshold T decreases, the middle equilibrium

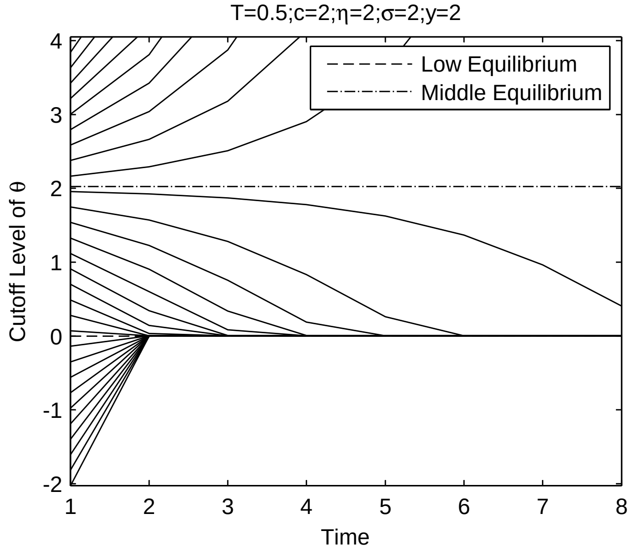

Figure 4.2: Best Response Dynamics of θ

cutoff $\tilde{\theta}_M$ increases. This would imply that players play less aggressively when the strength of government is smaller. Thus, we conclude that it is reasonable to exclude the middle cutoff equilibrium.

In addition the lowest equilibrium $\tilde{\theta}_L$ is obtained by best response dynamics from the lowest strategy. Thus the iterated elimination of dominated strategies eliminates all strategies, for which the regime would fall at lower states than $\tilde{\theta}_L$. It is the lower bound on all states, in which the regime would change. This also implies that if the function $A(\theta)$ is below the threshold T the only possible equilibrium is the one, in which no one revolts.

4.3.4 The Effect of the Public Signal

The posterior is increasing in the sense of first order stochastic dominance in the public signal. Thus increasing the signal moves $\tilde{x}(\theta)$ downwards and the function $A(\theta)$ upwards. The figure 4.3 illustrates the effect of public signal. For very small signal the peak of $A(\theta)$ is under the threshold T and there is only the highest equilibrium, in which no types participate. As the signal y rises $A(\theta)$ moves upwards and there appear two cutoff equilibria. The cutoff point of the lowest equilibrium is decreasing in the public signal.

Thus low public signal can lead to a situation, where nobody will revolt even though most of types are high and large part of the population would gain from the revolution. In addition the signal moves with the cutoff point $\tilde{\theta}$. The dot-and-dash line on the left represents the size of action, when every type greater then zero revolts. The line is the maximum size action that is attainable when all types that would benefit from revolution revolt. The intersection of the dot-and-dash with the threshold line T is smallest value the parameter Θ, for which there could be any revolution. As the public signal rises, the function $A(\theta)$ approaches the dot-and-dash line. So for high signal the outcome revolution will be successful for the same values of θ as in the complete information game.

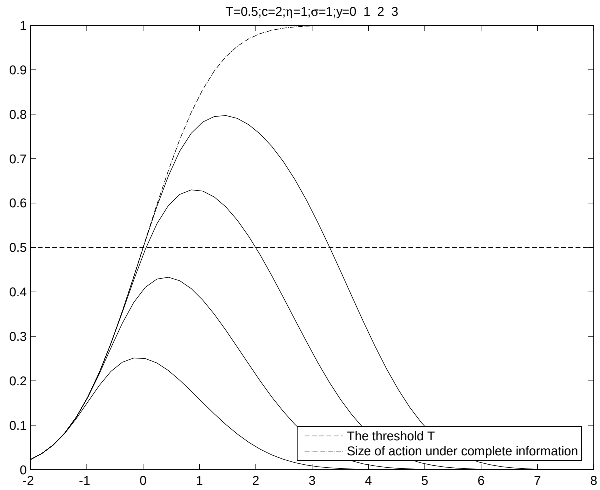

Figure 4.3: The Effect of the Public Signal

The effect of precision on the function $A(\theta)$ is also interesting. The figure 4.3 illustrates the influence of the precision on the function $A(\theta)$ and on the equilibria. If the public signal is uninformative, when there is only the private signal, the function $A(\theta)$ and is increasing and it always intersects T. After the introduction of a public signal, $A(\theta)$ gets the shape of a peak. As the precision rises the peak becomes sharper. As a we see on the picture for intermediate precisions the active equilibrium cease to exist for some intermediate values of precision.

4.4 Summary

In this chapter we introduced heterogeneity in the baseline model of collective action. When the payoffs of all people are well known in the society the game is of complete information and provided there is enough people for whom it is not dominant to abstain, there are two equilibria. One in which all types abstain and the second in which all positive type revolt.

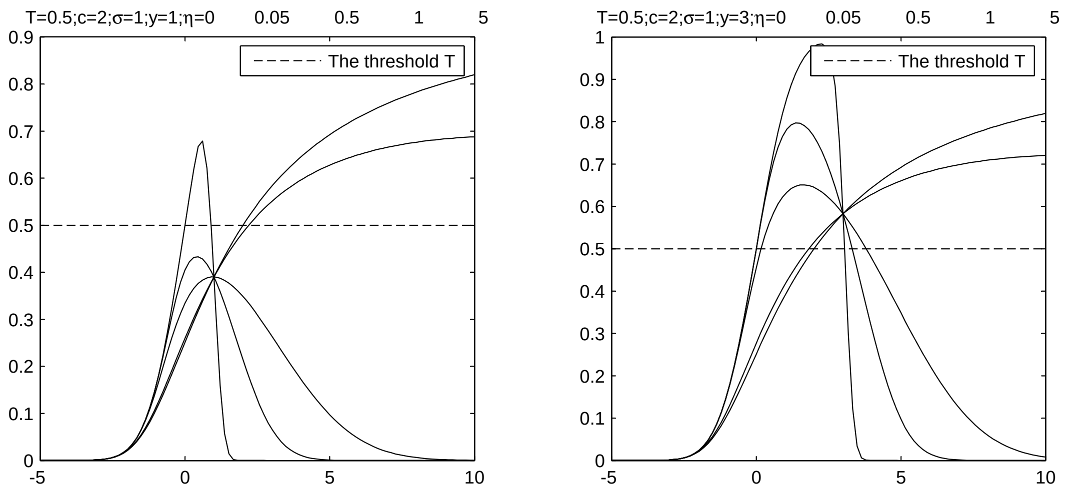

Figure 4.4: The Effect of the Precision of the Public Signal

When we introduced the idiosyncratic uncertainty about the exact distribution of payoffs the outcome of the game changes. We mentioned the fact that there are no states, for which the revolting would dominant for all types. In the language of Morris & Shin (2001) the game does not satisfy the limit dominance, one of the necessary conditions for the equilibrium uniqueness in the global game. As a consequence the self-fulfilling nature of the coordination problem remain. Yet, the introduction of the uncertainty still yields interesting results.

With the idiosyncratic signals, we identified equilibria in monotone strategies. Strategies, that are such that the types revolt if they are higher than some cutoff level and the regime falls if the threshold is higher than some cutoff level. When there was only private signal there was only one cutoff strategy in which some types revolted. When we added another public information there are from none to two equilibria in the monotone strategies in which some types revolted.

The uncertainty about higher order beliefs still causes that participation of some types is eliminated. The lowest cutoff point is the minimum type that will participate in the revolt. The public signal has an effect on the choices of the type. High signal induces more types to revolt and lowers the cutoff levels downward. The model, thus, provides a framework for a model of impact of informations. With private information only there is always an equilibrium with active participation, but low public signal can eliminate revolting completely, a feature that we did not see in the previous models.

We saw that the so called two-sided dominance cannot be dispensed if we want to get a unique prediction. Yet for the interpretation of payoffs, it is probably reasonable to assume only one-sided dominance. The multiplicity of equilibria is a both problem and advantage. The model is somehow incomplete as the outcome depends on exogenously set beliefs. On the other hand, through this optics we can explain sudden changes in outcomes without changes in the fundamentals. There could be long period peaceful period of no revolutionary activity. The revolting could eliminated for all types. But arrival of new informations can make the revolution possible. Change in the precision of information also has an effect on the the outcome of the game. We saw that especially for intermediate values of public signal, increased precision can also lead to making a revolt possible.

In the next chapter we briefly review some applications to revolutions as well as to currency crises.

Chapter 5

Applications

In this chapter we shortly review some applications of the model and explain its implications. Survey of more applications can be also found in Morris & Shin (2001).

5.1 Revolutions

Throughout the thesis we maintained the interpretation of model of the collective as a revolutionary attempt. Mesquita (2010) uses the model with heterogeneous payoffs as an explanation about the role terrorists during the revolution. He assumes that the dissidents can take a costly publicly visible action, of which success depends on the support of the people. The outcome of the action depends linearly on three factors: the amount invested in it by the dissidents, the public support and on a random component. In particular the dissident invests in the action e amount of effort and ϵ is the random component. Then the observed action is given as $e + \theta + \epsilon$.

Hence the outcome serves as a public signal. Thus the role of leaders in the revolution is not only to trigger the collective action by creating a focal point. Our model shows that the rationale for terrorists acts could be that they provide information. Since the outcome of terrorism depends positively on the public support, the action constitute publicly observable signal about anti-government sentiment in the population.

5.2 Currency Crises

A classic application of the global games methodology is model of currency crisis. A currency crisis is a sudden devaluation in a currency linked with an abandonment of fixed exchange rate. Investors can make profit during the currency crisis by short selling the currency with fixed rate and speculating on the devaluation of the currency. In many cases the devaluation in a currency is inevitable because the affected countries run inflationary policies. Obstfeld (1994) notes, however, that sometimes self-fulfilling expectations of investors can trigger short selling in large amounts and force a country to abandon the peg even if the peg could have been otherwise sustained.

Obstfeld (1986) argue that for some values of fundamentals crises is inevitable. For other values of fundamentals, however, both the currency crises and maintenance of the fixed exchange rate regime are possible in dependence on the expectations of the investors. Thus the currency crises contain similar self-fulling features as the model of revolution and have multiple equilibria. Morris & Shin (1998b) and Morris & Shin (1998a) applied the global game approach similar to that of section 3.3 to obtain unique prediction of the currency crises in dependence on the value of parameters. This allows us to make comparative statics exercise and see the role of the parameters.

We consider a stylized model of the currency attack. We assume that continuum of investors of mass one speculate on the devaluation of currency. Each investor can borrow one unit of currency and than sell the currency for an exchange rate e^0. After one period he one unit of currency back at an exchange rate e^1. So if the exchange is still fixed after one period he does not earn anything, but if the peg is abandoned and the currency depreciates, he will earn the differential between the exchange rates $e^0 - e^1$. Moreover we assume that carrying out this operation cost c unit of currency. Hence the benefit from an attack on currency is $B = e^0 - e^1 - c$.

We assume that central bank has resources to maintain the fixed exchange rate regime only if at most T units of currency are sold on the market. The T parametrizes the level of foreign currency reserves as well as determination of the bank to defend the peg. The model we just outlined corresponds to our model of collective action and we can apply our findings from previous chapters. We summarize the payoffs in the table 5.1

We consider the simplest case with uncertainty about the threshold T and only the idiosyncratic signals. The results follow those in 3.3.2. The bank will

	$M \geq T$	$M < T$
revolt	$e^0 - e^1 - c$	$-c$
abstain	0	0

Table 5.1: The Payoffs in the Model of Currency Crises

be able to defend the peg if for the parameter T:

$$\frac{e^0 - e^1 - c}{e^0 - e^1} \leq T \tag{5.1}$$

Speculating on abandonment of a peg is a almost safe bet. If the speculation works out the investors will earn the interest rate differential and if it does not the investors loose only their transaction costs. It was advocated (for example in Persaud (2009)) that making the transaction costs higher is a good policy against the currency crises. Indeed this is the case in our model. The lower is the transaction cost c, the larger will be the speculation attacks and the more often will be peg abandoned. When the transaction cost will be close to zero, the left hand side of 5.1 will be close to zero and the regime will collapse if $T < 1$. Thus even for currencies with strong fundamentals will be forced to leave the fixed exchange rate regime. Introducing a sand in the wheels might help to cut down on the number of self-fulfilling currency crises.

Atkeson (2000) argued that on financial markets investors closely observe public informations, hence the assumptions of only idiosyncratic signals are not viable and that the models must account for some restoration of common knowledge. There could be additional commonly known signals. We have shown that the cutoff level of the threshold T moves in the size of the public signals. Thus the change in the signals alone can trigger a currency crises. Moreover if the public informations are sufficiently precise the self-fulfilling nature of the crises is restored.

5.2.1 Extensions

It is often case that large investors of type of George Soros are highly visible and influence the small investors. Corsetti *et al.* (2004) investigated such situation in the framework of the the model of section 3.3. They assumed there is one large investor that can short sell α units of currency ($0 < \alpha < 1$) and there is continuum of small traders who can collectively short sell $1 - \alpha$ of currency. They found that attack by the large player induces the smaller investors to be more aggressive. The effect is caused not only because the large player lowers the amount of currency that the small players must short sell, but also because the action of large player conveys additional and commonly observed informations. The signal comes in the $\theta + \epsilon \geq y$ rather then in the form $\theta + \epsilon = y$ that tha signal had in our example. Similar situation to herding (Banerjee 1992) arises and players might heavily rely on the signal that the large player acts. The importance of the signal from the large player depends on the relative sizes of the variances of the signals of small and large investor.

Other authors investigated other sources of public information. Angeletos *et al.* (2006) considered a situation when a central bank endogenously set the transaction costs and Angeletos *et al.* (2007) such game is played repeatedly and players observe not only their signals but also know the peg was not abandoned. They found that public informations in such form again restore the multiplicity of equilibria.

Chapter 6

Conclusion

In this thesis we review theoretical literature about the coordination problem under uncertainty. We study a situation, in which agents choose between two actions, revolting and abstaining. If a sufficient part of the agents simultaneously choose the same action revolt, they can achieve a change in a regime that benefits them. Yet if only a small proportion of players revolt, those who revolt suffer a loss. Abstaining is an outside option that yields payoff zero. Hence each agent has to decide, whether to rely on that others will revolt as well. In the thesis we interpreted the situations as an attempt to overthrow a government, but the regime change could have a different meaning, for example an abandonment of fixed exchange rate.

The essence of the model is that the agents can reach the highest payoffs only if they coordinate. The choices of actions depend on the beliefs about what the other agents will do. Yet, many coherent beliefs are possible, so both of the actions can be justified as an equilibrium strategy. The outcomes are self-fulfilling, whatever outcome is believed to prevail, that outcome also happens. Moreover, we argued there is often a trade-off between safe and Pareto-efficient actions, so there is no clear criteria which actions to choose.

In the real world the agents seldom know certainly all the information about the situation, thus we reviewed the literature that introduced the uncertainty into the model of coordination. We described especially the so called global game, the coordination game of incomplete information in which agents received different but correlated signals about the state. As a consequence there is uncertainty not only about the parameters of the game, but also about the information of other agents. We showed that the approach has two advantages. For one thing it turns out that this setting often leads to unique equilibrium and it thus enables us to examine the effect of the parameters. For the other thing the setting provides us with framework, in which we can discuss the role of information.

We consider several variants of the uncertainty and compared the effects. We consider model, in which each player receives idiosyncratic signal, the priors are uninformative and both action were dominated at some states. The model is a classic example of the global game. In this model the multiplicity of the equilibria disappeared. Great advantage is the unique prediction of the model and the possibility of making a comparative exercise.

We show that such model could be used to explain a currency crises. We showed that throwing some sand in the wheels of financial markets, raising the transaction costs of financial operations, would deter the agents from speculating against currency and reduce the occurrence of the currency crises

We also considered that players receive additional signal that is shared by all players. We found that if such signal was precise relatively to the private signal the multiplicity of equilibria was restored. We described how the result highlights the role of assumptions behind the global game. With sufficiently precise public signal the fact that the both actions are not dominated is commonly believed, whereas in the previous case there was always great uncertainty about the beliefs of other players, which led to elimination of some equilibria.

Lastly we consider a version of the model, in which only one action was dominated at some states. As a consequence the self-fulfilling nature of the equilibria was restored. We demonstrated that the public information affects the outcome of the game. One source of such information in political regime situation could be a publicly observable actions of terrorist. The model thus provides rationale for a terrorist act, which provides commonly observable information about the fundamentals. Such signal can make the revolution possible by informing the agents that the anti-government sentiment in the society is sufficiently high.

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Appendix A

Appendix

A.1 Bayesian Updating

Below we derive the posterior distribution that we have used throughout the thesis. We find the posterior for the random variable Θ with normal prior after observing a noisy signal about Θ with normally distributed noise. The derivation follows DeGroot (2005).

We assume that Θ and Y are distributed according:

$$\Theta \sim \mathcal{N}(x, \sigma), \quad Y \sim \mathcal{N}(\Theta, \eta) \tag{A.1}$$

We denote $f_\Theta(\theta|y)$ the probability density function of Θ at $\Theta = \theta$ conditional on observing that $Y = y$. $f_\Theta(\theta)$ is the density function for Θ and $f_Y(y|\theta)$ is the density for Y conditional on realization of $\Theta = \theta$. We plug the normal densities into the rule for finding conditional probability density functions:

$$\begin{aligned} f_\Theta(\theta|y) &= \frac{f_Y(y|\theta) f_\Theta(\theta)}{\int\limits_{-\infty}^{\infty} f_Y(y|t) f_\Theta(t) dt} \propto f_Y(y|\theta) f_\Theta(\theta) \\ &\propto \exp(-\eta(y-\theta)^2) \exp(-\sigma(\theta - x)^2) \\ &\propto \exp(-\eta y^2 + 2\eta y\theta - \eta\theta^2 - \sigma\theta^2 + 2\sigma\theta x - \sigma x^2) \end{aligned}$$

In the following equations we rearrange the last term so that all θ are in quadratic term and leave the rest that is not a function of θ out.

$$\begin{aligned}
f_\Theta(\theta|y) &\propto \exp(-\eta y^2 + 2\eta y\theta - \eta\theta^2 - \sigma\theta^2 + 2\sigma\theta x - \sigma x^2) \\
&\propto \exp(-(\eta+\sigma)\theta^2 + 2(\sigma x + \eta y)\theta - \eta y^2 - \sigma x^2) \\
&\propto \exp\Bigg(-(\eta+\sigma)\Bigg(\theta^2 + 2\left(\frac{\sigma}{\eta+\sigma}x + \frac{\eta}{\eta+\sigma}y\right)\theta + \left(\frac{\sigma}{\eta+\sigma}x + \frac{\eta}{\eta+\sigma}y\right)^2 \\
&\quad - \left(\frac{\sigma}{\eta+\sigma}x + \frac{\eta}{\eta+\sigma}y\right)^2\Bigg) - \eta y^2 - \sigma x^2\Bigg) \\
&\propto \exp\left(-(\eta+\sigma)\left(\theta - \left(\frac{\sigma}{\eta+\sigma}x + \frac{\eta}{\eta+\sigma}y\right)\right)^2\right) \\
&\quad \times \exp\left((\eta+\sigma)\left(\frac{\sigma}{\eta+\sigma}x - \frac{\eta}{\eta+\sigma}y\right)^2 - \eta y^2 - \sigma x^2\right) \\
&\propto \exp\left(-(\eta+\sigma)\left(\theta - \left(\frac{\sigma}{\eta+\sigma}x + \frac{\eta}{\eta+\sigma}y\right)\right)^2\right)
\end{aligned}$$

The last term is proportional to the density of normal distribution. Since $f_\Theta(\theta|y)$ is probability distribution function, it must integrate to one and thus the multiplicative constant must be the same as for the density of normal distribution. Thus the posterior distribution of Θ is:

$$\mathcal{N}\left(\frac{\sigma}{(\eta+\sigma)}x + \frac{\eta}{(\eta+\sigma)}y, \frac{1}{\eta+\sigma}\right) \tag{A.2}$$

After the player receives the private signal x has the same posterior as the prior in A.1, so after he observes the public signal Y he further updates the posterior, so the posterior distribution is given by A.2.

A.2 Cutoff Equilibria with Heterogeneous Payoffs

Below we show the proof for the claim in section 4.

Claim A.1. *If the players receive only private signals, $A(\theta)$ is increasing. If there is public signal available, $A(\theta)$ is single peaked and* $\lim_{\theta\to-\infty} A(\theta) = \lim_{\theta\to\infty} A(\theta) = 0$

Proof. The function $A(\theta)$ is defined as $1 - \Phi\Big(\sqrt{\sigma}(\tilde{x}(\theta) - \theta)\Big)$. To examine whether it is increasing or not we differentiate it.

$$\frac{dA}{d\theta} = -\sqrt{\sigma}\phi\Big(\sqrt{\sigma}(\tilde{x}(\theta) - \theta)\Big)(\tilde{x}'(\theta) - 1)$$

$A(\theta)$ is increasing iff:

$$\frac{dA}{d\theta} = -\sqrt{\sigma}\phi\Big(\sqrt{\sigma}(\tilde{x}(\theta) - \theta)\Big)(\tilde{x}'(\theta) - 1) > 0$$

Since $\phi(\cdot)$ is positive, $A(\theta)$ is increasing iff $\tilde{x}'(\theta) < 1$. The function $\tilde{x}(\theta)$ is defined implicitly as $\tilde{x}$ that solves.

$$\left(1 - \Phi\left(\eta + \sigma\left(\theta - \frac{\sigma}{\eta+\sigma}\tilde{x} - \frac{\eta}{\eta+\sigma}y\right)\right)\right)(\tilde{x} + c) = c$$

In what follows we will denote $\phi(\sqrt{\eta+\sigma}(\theta - \frac{\sigma}{\eta+\sigma}\tilde{x} - \frac{\eta}{\eta+\sigma}y))$ and $\Phi(\sqrt{\eta+\sigma}(\theta - \frac{\sigma}{\eta+\sigma}\tilde{x} - \frac{\eta}{\eta+\sigma}y))$ as f and F. By implicit differentiation we find that the derivative of $\tilde{x}(\theta)$ is:

$$\tilde{x}'(\theta) = -\frac{-f(\tilde{x} + c)}{\frac{\sigma}{\eta+\sigma}f(\tilde{x} + c) + 1 - F} \tag{A.3}$$

Since $A(\theta)$ is increasing iff $\tilde{x}'(\theta) < 1$, we plug the derivative into this condition. After multiplying both sides by the positive denominator we get that $A(\theta)$ is increasing iff:

$$f(\tilde{x} + c) < \frac{\sigma}{\eta+\sigma}f(\tilde{x} + c) + 1 - F$$

After subtracting $\frac{\sigma}{\eta+\sigma}f(\tilde{x} + c)$ from both sides and dividing by $f(\tilde{x} + c)$ we get that this is equivalent to:

$$\frac{\eta}{(\eta+\sigma)} < \frac{1 - F}{f(\tilde{x} + c)} \tag{A.4}$$

When there is only private information the public signal is completely uninformative and its precision is zero. Therefore to obtain the slope of $A(\theta)$ in this special case we substitute $\eta = 0$ into the inequality $A.4$. The condition is always satisfied because the right hand side of the inequality is strictly positive and therefore the function $A(\theta)$ is increasing.

When the public signal is present, the left hand side of the inequality $A.4$ is greater than zero. We show that right hand side is decreasing in θ and that it crosses the left hand side once. This implies that the function $A(\theta)$ is single peaked.

First we show that the argument inside the normal cdf and pdf on the left hand side of the inequality A.4 is increasing in θ, i.e. that:

$$\sqrt{\eta+\sigma}\left(\theta - \frac{\sigma}{\eta+\sigma}\tilde{x}(\theta) - \frac{\eta}{\eta+\sigma}y\right) \tag{A.5}$$

is increasing in θ. We differentiate the expression and plug in for the derivative of $\tilde{x}(\theta)$ from A.3. So the term is increasing in θ iff:

$$\begin{aligned}
1 - \frac{\sigma}{\eta+\sigma}x'(\theta) &> 0 \\
1 - \frac{\sigma}{\eta+\sigma}\left(-\frac{-f(\tilde{x}+c)}{\frac{\sigma}{\eta+\sigma}f(\tilde{x}+c)+1-F}\right) &> 0 \\
\frac{\sigma}{\eta+\sigma}f(\tilde{x}+c)+1-F &> \frac{\sigma}{\eta+\sigma}f(\tilde{x}+c) \\
1-F &> 0
\end{aligned}$$

The last term is indeed true, so A.5 is increasing in θ. Because the normal distribution is log concave (see Bagnoli & Bergstrom (2005)) the ratio $\frac{1-\Phi(\cdot)}{\phi(\cdot)}$ is decreasing in its arguments. $\frac{1-F}{f}$ on the left hand side of A.4 is composed function of the increasing term A.5 and the decreasing ratio of normal cdf and pdf $\frac{1-\Phi(\cdot)}{\phi(\cdot)}$, hence it is decreasing. Moreover since $\tilde{x}(\theta)+c$ is positive and increasing, $\frac{1-F}{f(\tilde{x}+c)}$ is decreasing. Next, we investigate the limits of $\frac{1-F}{f(\tilde{x}+c)}$. We note that:

$$\lim_{\theta\to-\infty}\tilde{x}(\theta)=0, \quad \lim_{\theta\to\infty}\tilde{x}(\theta)=\infty \tag{A.6}$$

As θ goes to $-\infty$:

$$\lim_{\theta\to-\infty}\sqrt{\eta+\sigma}\left(\theta - \frac{\sigma}{\eta+\sigma}\tilde{x}(\theta) - \frac{\eta}{\eta+\sigma}y\right) = -\infty$$

Because the arguments inside of the normal cdf and pdf in F and f go to $-\infty$, F and f go to zero, so we have that:

$$\lim_{\theta\to-\infty}\frac{1-F}{f(\tilde{x}(\theta)+c)} = \infty$$

In addition the term $\frac{1-F}{f}$ is decreasing and greater than zero, so it has a limit. As θ goes to infinity:

$$\lim_{\theta\to\infty}\frac{1-F}{f(x(\theta)+c)}=\lim_{\theta\to-\infty}\frac{1-F}{f}\times\lim_{\theta\to-\infty}\frac{1}{x(\theta)+c}=\lim_{\theta\to-\infty}\frac{1-F}{f}\times 0=0$$

To sum it up, the term on the right hand side of the inequality A.4 is decreasing in θ and goes from infinity to zero. Thus for small θ the inequality A.4 holds and $A(\theta)$ is increasing and for high θ the function $A(\theta)$ is decreasing. Next we look at the limits of $A(\theta)$.

$$\begin{aligned}\lim_{\theta\to-\infty}A(\theta)&=\lim_{\theta\to-\infty}1-\Phi\Big(\sqrt{\sigma}(\tilde{x}(\theta)-\theta)\Big)\\&=1-\Phi(\infty)=0\end{aligned}$$

In the end we show that as θ goes to infinity $A(\theta)$ goes to zero. First we look at the limit of $x'(\theta)$:

$$\lim_{\theta\to\infty}x'(\theta)=\lim_{\theta\to\infty}\frac{1}{\frac{\sigma}{\eta+\sigma}+\frac{1-F}{f(x(\theta)+c)}}=\frac{1}{\frac{\sigma}{\eta+\sigma}}=\frac{\eta+\sigma}{\sigma}$$

Then:

$$\lim_{\theta\to\infty}x(\theta)-\theta=\lim_{\theta\to\infty}\theta\left(\frac{x(\theta)}{\theta}-1\right)=\lim_{\theta\to\infty}\theta\times\lim_{\theta\to\infty}(x'(\theta)-1)=\frac{\eta}{\sigma}\lim_{\theta\to\infty}\theta=\infty$$

The second equality follows from l'Hospital's rule.

$$\lim_{\theta\to\infty}A(\theta)=\lim_{\theta\to\infty}1-\Phi\Big(\sqrt{\sigma}\big(\tilde{x}(\theta)-\theta\big)\Big)=1-\Phi(\infty)=0$$

Thus we found, that in the case with public signal the function $A(\theta)$ is single-peaked and $\lim_{\theta\to-\infty}A(\theta)=\lim_{\theta\to\infty}A(\theta)=0$.

□

Master Thesis Proposal

Institute of Economic Studies
Faculty of Social Sciences
Charles University in Prague

Author:	**Bc. Tomáš Fiala**	Supervisor:	PhDr. Martin Gregor PhD
E-mail:	tomes.fiala@seznam.cz	E-mail:	gregor@fsv.cuni.cz
Phone:	725558532	Phone:	
Specialization:	Economic Theory	Defense Planned:	June 2012

Proposed Topic:

Global Games

Topic Characteristics:

Multiple equilibria are common in games where players need to coordinate. For example, it is often observed that currency attacks are triggered even without change in economic fundamentals. Obsfeld (1996) builds a model of currency attack which features multiple equilibria and sudden currency attack might be explained by a change in the beliefs. In the coordination games the outcome does not depend only on the beliefs of each player about the fundamentals but also on the beliefs about the beliefs of other player and so on to infinity. Rubinstein (1989) showed that even slight absence of such higher order beliefs might lead to very qualitatively different results. Morris and Shin (1998) consider variation of the Obsfeld`s model. They assume that the fundamentals are not known perfectly and model the belief formation of the players explicitly. They showed that even small degree of uncertainty leads to unique equilibrium. This approach of Morris and Shin(1998) (originally by Carlsson, H., and E. van Damme (1993)) is applicable to wider class of problems in which players need to coordinate on their actions (e.g. debt pricing, bank runs).

Another important situation, in which coordination is vital is a revolution. The outcome of a revolution depends also depends on the beliefs of people on others` participation. If noone beliefs that government will fall no revolution will take place and vice versa. Such multiplicity of equilibria leaves much the process unexplained and constitutes poor framework for analysis of the causes of sudden political change. The above outlined aproach of Global Games might provide good way how to model equilibrium selection in such situations. In the thesis we will discuss how the framework of Global games is relevant to the study of political change and then consider an application.

We will focus on setting of sequential move game (as in Angeletos, G.-M., C. Hellwig, and A. Pavan (2006) or Edmond, C.(2007)). This will allow us examine the role of learning and signaling in the coordination problem. Such approach might be motivated by answering topical question whether the spread of more efficient means of communication (such as the internet or uncensored cable TV) might help to coordinate actions to overthrow a political regime.

Hypotheses:

1. Can cheap talk or costly signals by government strengthen its position?
2. Does a large player have significant impact?
3. Can learning and signaling among players lead to better coordination?

Methodology:

This thesis will review the literature on global games and the role of common knowledge in coordination problems and discuss its applicability to the problem of the political regime change. Then we will use the apparatus of global games and game theory generally to model it.

We will consider slight variations in the assumption of the basic framework of Bueno De Mesquita, E. (2010). We will try to extend it to dynamic case, to allow for learning. We will focus on signaling by government (as in Edmond, C.(2007)) and the role of large player's signals (as in Corsetti, G. at al. (2004)). We will attempt to obtain analytical results if possible and/or use a numerical simulation to answer the questions.

Outline:

1) Literature review
    a. Current models of revolutions
    b. Game theoretic aprroach
    c. Information economics, Global games, Higher order beliefs
2) How can global games contribute to the study of revolution?
    a. Using Game Theory to study revolutions
    b. Multiplicity of equlibria and equilibrium selection
3) Applications of Global Games Approach to Political Change
    a. Basic Model
    b. Extension
    c. Comparative analysis
    d. Empirical testability

Core Bibliography:

Angeletos, G.-M., C. Hellwig, and A. Pavan (2006): Signaling in a global game: Coordination and policy traps, Journal of Political Economy, 114(3), 452-484. 1, 3, 5, 10, 11

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Author

Supervisor