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The Instrumental Weighted Variables. Part II. $\sqrt{n}$ - consistency

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The Instrumental Weighted Variables. Part II. $\sqrt{n}$ - consistency

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Abstract:
The definition of Instrumental Weighted Variables (IWV) (which is a robust version of the classical Instrumental Variables) and conditions for the weak consistency as given in the Part I of this paper are recalled. The reasons why the classical Instrumental Variables were introduced as well as the idea of implicit weighting the residuals (firstly employed by the Least Weighted Squares, see Víšek (2000)) are also recalled. Then $\sqrt{n}$ -consistency of all solutions of the corresponding normal equations is proved.

Keywords: Robustness, instrumental variables, implicit weighting, $\sqrt{n}$ -consistency of estimate by means of instrumental weighted variables

AMS classification: 62F35, 62J05

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INTRODUCTION

The paper continues in studies of Víšek (2006b). That it why we recall reasons for introducing the *Instrumental Weighted Variables* as well as for employing the idea of implicit weighting residuals, as firstly used in Víšek (2000), only briefly. Nevertheless, we will do it in a way to make the paper self-contained.

Let N denote the set of all positive integers, R the real line and R^p the p-dimensional Euclidean space. We are going to consider the linear regression model given as

$$Y_i = X_i'\beta^0 + e_i = \sum_{j=1}^{p} X_{ij}\beta_j^0 + e_i, \quad i = 1, 2, ..., n. \tag{1}$$

Without loss of generality we may assume that $\beta^0 = 0$, but $\beta - \beta^0$ is written instead of just β when we deal with β from the neighborhood of the true value β^0. The following conditions are assumed to be fulfilled.

C1 *The sequence* $\left\{(X_i^T, e_i)^T\right\}_{i=1}^{\infty}$ *is sequence of independent and identically distributed* $p+1$*-dimensional random vectors (i.i.d. r.v.'s) with absolutely continuous distribution function* $F_{X,e}(x,v)$*. Moreover,* $I\!E\left\{(X_1^T, e)^T \cdot (X_1^T, e)\right\}$ *is positive definite matrix and the density* $f_{e|X}(v|X_1 = x)$ *is uniformly in* x *bounded in* v*, say by* U_e*.*

$F_X(x)$ and $F_e(v)$ ($f_X(x)$ and $f_e(v)$) will stay for the marginals of $F_{X,e}(x,v)$ (and their densities, respectively). (Throughout the paper all vectors will be assumed the column ones.) Finally, notice please that $f_e(v) = I\!E_x f_e(v|X_1 = x) \le I\!E_x U_e = U_e$.

We shall study the model with intercept, i.e. we assume that the first coordinate of explanatory variables X_i is degenerated and equal to 1.

ESTIMATING BY MEANS OF INSTRUMENTAL VARIABLES

The most frequently used estimator of the regression coefficients β^0 of the "true" underlying model is the *(Ordinary) Least Squares* $\hat\beta^{(OLS,n)}$. Due to the fact that

$$\hat\beta^{(OLS,n)} = \beta^0 + \left(\frac{1}{n}\sum_{k=1}^{n} X_k X_k'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i e_i \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i e_i = I\!E X_1 e_1 \quad \text{in probability,} \tag{2}$$

one easy verifies that the violation of orthogonality condition $I\!E\{e_i|X_i\} = 0$ implies inconsistency of the *(Ordinary) Least Squares* (where due to **C1** $\frac{1}{n}\sum_{k=1}^{n} X_k X_k'$ is, starting with some n_0 (say), positive definite almost surely).

One of the best known example of the situations when the orthogonality condition fails, was discussed in the first part of these three papers (Víšek (2006b)). We are going to recall another famous example justifying employment of the method of instrumental variables. The model, we will consider, is not a special case of (1). When we arrive at (4), we can easy verify that the rows are correlated and we have to use a transformation of Cochrane-Orcutt (see Cochrane, Orcutt (1949)) or Prais-Winsten type (see Prais, Winsten (1954)) to fulfil assumptions of the model (1). However, it would bring a large notational complexity (although it represents only a technical problem) and it may obscure the idea of the next example. So let us consider (with a bit of freedom from the rigor) the model with lagged explanatory variables. Assume the simplest one, with the geometric structure of coefficients, i. e.

$$Y_t = \gamma \sum_{j=1}^{\infty} \lambda^{j-1} x_{t-j+1} + e_t, \quad t = ..., -1, 0, 1, 2, ..., T \tag{3}$$

with $I\!\!E e_t = 0$ and $I\!\!E e_t^2 = \sigma^2 \in (0, \infty)$. Clearly, we are not able to estimate coefficients γ and λ, so writing model for $t-1$

$$Y_{t-1} = \gamma \sum_{j=1}^{\infty} \lambda^{j-1} x_{t-j} + e_{t-1},$$

multiplying it by λ and subtracting from (3), we obtain

$$Y_t = \lambda Y_{t-1} + \gamma x_t + e_t - \lambda e_{t-1} = \lambda Y_{t-1} + \gamma x_t + u_t. \tag{4}$$

Now, the "explanatory" variable Y_{t-1} is correlated with the error term u_t and then (2) indicates that OLS estimate of regression coefficients of model (4) is inconsistent.

Another frequently presented example considers the situation when the explanatory variables are measured with a random error, see Judge et al. (1985) or Víšek (1998), (2006b).

The classical econometrics solve such situations *usually* by means of the *Method of Instrumental Variables.*

Definition 1 *For any sequence of random vectors* $\{Z_i\}_{i=1}^{\infty} \subset R^p$ *the solution(s) of the (vector) equation*

$$\sum_{i=1}^{n} Z_i \left(Y_i - X_i^T \beta \right) = 0 \tag{5}$$

will be called the estimator obtained by means of the method of Instrumental Variables (or Instrumental Variables, for short) and denoted by $\hat{\beta}^{(IV,n)}$.

The method became at the end of the last century more or less a standard tool in many case studies of panel data since the correlation of explanatory variables and disturbances frequently appeared. Papers exploring the best way of the selecting the instruments for explanatory variables established useful, easy implemented results, see e.g. Arellano, Bond (1991), Arellano, Bover (1995) or Sargan (1988) (and for examples of implementation see for SAS - Der and Everitt (2002), for R and S-PLUS - Fox, J. (2002)).

RECALLING THE LEAST WEIGHTED SQUARES

Let us enlarge a bit the notations. Let us denote for any $\beta \in R^p$ by $r_i(\beta) = Y_i - X_i'\beta$ the i-th residual and by $r_{(h)}^2(\beta)$ the h-th order statistic among the squared residuals. To be more explicite, we have

$$r_{(1)}^2(\beta) \le r_{(2)}^2(\beta) \le \; ... \; \le r_{(n)}^2(\beta). \tag{6}$$

Then the *Least Weighted Squares* can be defined as follows (see Víšek (2000), see also (2002b, c)):

$$\hat{\beta}^{(LWS,n,w)} = \underset{\beta \in R^p}{\arg\min} \sum_{i=1}^{n} w_i r_{(i)}^2(\beta) \tag{7}$$

where $w_i, i = 1, 2, ..., n$ are weights[1]. They are usually generated by a weight function with the following properties[2]:

C2 *Weight function* $w : [0, 1] \to [0, 1]$ *is absolutely continuous and nonincreasing, with the derivative* $w'(\alpha)$ *bounded from below by* $-L$, $w(0) = 1$.

Then put $w_i = w\left(\frac{i-1}{n}\right)$. Following Hájek, Šidák (1967) for any $i \in \{1, 2, ..., n\}$ let us denote by $\pi(\beta, i)$ the rank of the i-th residual. It means that $\pi(\beta, i) = j \in \{1, 2, ..., n\}$ iff $r_i^2(\beta) = r_{(j)}^2(\beta)$ (notice that $\pi(\beta, i)$ is r.v.). Then we have

$$\hat{\beta}^{(LWS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w\left(\frac{\pi(\beta, i) - 1}{n}\right) r_i^2(\beta). \quad (8)$$

It is straightforward to show that the *Least Weighted Squares* are solution of *normal equations*

$$I\!N E_{X,n}(\beta) = \sum_{i=1}^{n} w\left(\frac{\pi(\beta, i) - 1}{n}\right) X_i \left(Y_i - X_i'\beta\right) = 0, \quad (9)$$

see Víšek (2006b).

INSTRUMENTAL WEIGHTED VARIABLES

The inconsistency of the *Ordinary Least Squares* which is due to the failure of the orthogonality condition (as we recalled it in INTRODUCTION), takes place generally also for the *Least Weighted Squares*. That is why we define an estimator which will be an analogy of the estimator obtained by the *Method of Instrumental Variables* but which will weight down the residuals of those observations which seem to be atypical. For complex discussion see Hampel et al. (1986) or Rousseeuw and Leroy (1987).

Definition 2 *For any sequence of random vectors* $\{Z_i\}_{i=1}^{\infty} \subset R^p$ *the solution(s) of the (vector) equation*

$$I\!N E_{Z,n}(\beta) = \sum_{i=1}^{n} w\left(\frac{\pi(\beta, i) - 1}{n}\right) Z_i \left(Y_i - X_i'\beta\right) = 0 \quad (10)$$

will be called the Instrumental Weighted Variables estimator and denoted by $\hat{\beta}^{(IWV,n,w)}$.

Remark 1 *The elements of the sequence* $\{Z_i\}_{i=1}^{\infty}$ *are usually called instruments. Without loss of generality we may assume that* $Z_{i1} = 1$ *and* $I\!E Z_{ij} = 0, j = 2, 3, ..., p$ *and* $i = 1, 2, ...$ *We do not lose generality firstly, due to the fact that* $Z_{i1} = 1$ *represents constants and hence they cannot be correlated with disturbances (in fact we have then* $Z_{i1} = X_{i1}$*). Secondly, what concerns the assumption that* $I\!E Z_{ij} = 0, j = 2, 3, ..., p$*, if it would not be fulfilled, we can "move"* $I\!E Z_{ij}$ *into the intercept of the original model (1).*

For any $\beta \in R^p$ the distribution of the absolute value of residual will be denoted $F_\beta(v)$. In other words,

$$F_\beta(v) = P(|Y_1 - X_1'\beta| < v) = P\left(\left|e_1 - X_1'\left(\beta - \beta^0\right)\right| < v\right). \quad (11)$$

[1]See also Čížek (2002) where the estimator is called the *Smoothed Least Trimmed Squares.*

[2]Compare Hájek, Šidák (1967).

Similarly, for any $\beta \in R^p$ the empirical distribution of the absolute value of residual will be denoted $F_\beta^{(n)}(v)$. It means that, denoting the indicator of a set A by $I\{A\}$, we have

$$F_\beta^{(n)}(v) = \frac{1}{n}\sum_{j=1}^{n} I\{|r_j(\beta)| < v\} = \frac{1}{n}\sum_{j=1}^{n} I\left\{|e_j - X_j'\beta| < v\right\}$$

$$= \frac{1}{n}\sum_{j=1}^{n} I\left\{\omega \in \Omega \,:\, |e_j(\omega) - X_j'(\omega)\beta| < v\right\}. \tag{12}$$

It is straightforward that then (for details see Víšek (2006b))

$$F_\beta^{(n)}(|r_i(\beta)|) = \frac{\pi(\beta, i) - 1}{n}$$

and so (10) can be written as

$$\sum_{i=1}^{n} w\left(F_\beta^{(n)}(|r_i(\beta)|)\right) Z_i\left(Y_i - X_i'\beta\right) = 0. \tag{13}$$

CONSISTENCY OF THE INSTRUMENTAL WEIGHTED VARIABLES

We will need also the following notation. For any $\beta \in R^p$ the distribution of the product $\beta' Z X' \beta$ will be denoted $F_{\beta' Z X' \beta}(u)$, i. e.

$$F_{\beta' Z X' \beta}(u) = P(\beta' Z X' \beta < u) \tag{14}$$

and similarly as in previous, the corresponding empirical distribution will be denoted $F_{\beta' Z X' \beta}^{(n)}(u)$, so that

$$F_{\beta' Z X' \beta}^{(n)}(u) = \frac{1}{n}\sum_{j=1}^{n} I\left\{\beta' Z_j X_j' \beta < u\right\} = \frac{1}{n}\sum_{j=1}^{n} I\left\{\omega \in \Omega \,:\, \beta' Z_j X_j' \beta < u\right\}. \tag{15}$$

For any $\zeta \in R^+$ and any $a \in R$ put

$$\gamma_{\zeta,a} = \sup_{\|\beta\|=\zeta} F_{\beta' Z X' \beta}(a). \tag{16}$$

Notice please that due to the fact that the surface of the ball $\{\beta \in R^p, \|\beta\| = \zeta\}$ is compact, there is $\beta_\gamma \in \{\beta \in R^p, \|\beta\| = \zeta\}$ so that

$$\gamma_{\zeta,a} = F_{\beta_\gamma' Z X \beta_\gamma}(a). \tag{17}$$

For any $\zeta \in R^+$ let us denote

$$\tau_\zeta = -\inf_{\|\beta\|\le\zeta} \beta' I\!E\left[Z_1 X_1' \cdot I\{\beta' Z_1 X_1' \beta < 0\}\right]\beta. \tag{18}$$

Notice please that $\tau_\zeta \geq 0$ and that again due to the fact that the ball $\{\beta \in R^p, \|\beta\| \leq \zeta\}$ is compact, the infimum is finite, and hence there is a $\tilde{\beta} \in \{\beta \in R^p, \|\beta\| \leq \zeta\}$ so that

$$\tau_\zeta = -\tilde{\beta}' I\!E\left[Z_1 X_1' \cdot I\{\tilde{\beta}' Z_1 X_1' \tilde{\beta} < 0\}\right]\tilde{\beta}. \tag{19}$$

C3 *The instrumental variables* $\{Z_i\}_{i=1}^{\infty} \subset R^p$ *are independent and identically distributed with distribution function* $F_Z(z)$. *Moreover, they are independent from the sequence* $\{e_i\}_{i=1}^{\infty}$. *Further, the joint distribution function* $F_{X,Z}(x,z)$ *is absolutely continuous,* $I\!E\left\{w(F_{\beta^0}(|e_1|))Z_1X_1^T\right\}$ *as well as* $I\!E Z_1 Z_1^T$ *are positive definite (one can compare C3 with Víšek (1998) where we considered instrumental* M*-estimators and the discussion of assumptions for* M*-instrumental variables was given) and there is* $q > 1$ *so that* $I\!E\{\|Z_1\| \cdot \|X_1\|\}^q < \infty$. *Finally, there is* $a > 0$, $b \in (0,1)$ *and* $\lambda > 0$ *so that*

$$a \cdot (b - \gamma_{\lambda,a}) \cdot w(b) > \tau_\lambda \tag{20}$$

for $\gamma_{\lambda,a}$ *and* τ_λ *given by (27) and (38).*

Remark 2 *Let us briefly discuss assumptions we have made. Let us recall that the Least Squares* ($\beta^{(LS,n)}$) *are optimal only under normality of disturbances. Here the optimality means that they reach the lower Rao-Cramer bound (in multivariate Rao-Cramer lemma we consider the ordering of the covariance matrices in the sense of ordering the positive definite matrices).On the other hand, a small departure from normality may cause (and usually does) a large decrease of efficiency (see e.g. Fisher (1920), (1922)). So, without the assumption of normality of disturbances* $\hat{\beta}^{(LS,n)}$ *is much worse, in fact they are the best unbiased estimator only in the class of linear unbiased estimators, for a discussion showing that restriction on linear estimators can be drastic see Hampel et al. (1986). Sometimes, however we may meet with the statement that we do not need necessarily the normality of disturbances, just because* $\hat{\beta}^{(LS,n)}$ *is still (without normality) the best unbiased estimator in the class of linear unbiased estimators. And the restriction on the class of linear unbiased estimators is justified by a claim that we have to restrict ourselves on the class of linear estimaors, as in the the class of linear unbiased estimators, the estimators are scale- and regression-eqivariant. Let us recall that having denoted* $M(n,p)$ *the set of all matrices of type* $(n \times p)$ *and recalling that the estimator* $\hat{\beta}$ *can be considered as a mapping*

$$\hat{\beta}(Y,X): \ M(n,p+1) \to R^p,$$

the estimator $\hat{\beta}$ *of* β^0 *is called scale-equivariant, if for any* $c \in R^+, Y \in R^n$ *and* $X \in M(n,p)$ *we have*

$$\hat{\beta}(cY,X) = c\hat{\beta}(Y,X)$$

and regression-equivariant if for any $b \in R^p, Y \in R^n$ *and* $X \in M(n,p)$

$$\hat{\beta}(Y + Xb, X) = \hat{\beta}(Y,X) + b.$$

But, there are a lot of nonlinear estimators which are scale- and regression-equivariant. In the regression framework, the estimators as the Least Median of Squares, the Least Trimmed Squares or the Least Weighted Squares can serve as examples (for an interesting discussion of this topic see again Hampel et al. (1986), and also Bickel (1975) or Jurečková and Sen (1993)).)

Since LWS are also based on L_2*-metric, we guess that they are approximately optimal for finite sample sizes under the (approximative) normality of disturbances, for some hint consult Mašíček (2003). As the present proposal of robustified instrumental variables is based on the same metric (due to the normal equations (10)), we can expect that the estimate can be approximately optimal under (approximative) normality of disturbances. But then our assumptions seem to be quite acceptable.*

The only assumption which deserve further discussion is the assumption (41). We are going to show that it is a restriction on the weight function w. Let us return to (27) (or to (29)). We have

$$\gamma_{\lambda,a} = F_{\beta_\lambda^T Z X^T \beta_\lambda}(a) = P\left(\beta_\lambda^T Z_1 X_1^T \beta_\lambda \leq 0\right) + P\left(0 < \beta_\lambda^T Z_1 X_1^T \beta_\lambda \leq a\right).$$

If we assume for a while $Z_j = X_j$, for any fix $\lambda \in R^+$ we have

$$\lim_{a\to\infty} F_{\beta_\gamma^T X X^T \beta_\gamma}(a) = 0 \tag{21}$$

but for $\gamma_{\lambda,a}$ we have (again for fix $\lambda \in R^+$)

$$\lim_{a\to\infty} F_{\beta_\gamma^T Z X^T \beta_\gamma}(a) = P\left(\beta_\lambda^T Z_1 X_1^T \beta_\lambda \leq 0\right). \tag{22}$$

On the other hand, for any $a > 0$ we have

$$\gamma_{\lambda,a} < 1. \tag{23}$$

Now let us turn to τ_λ. As

$$I\!E\left|\beta^T Z_1 X_1^T \beta\right| \leq \|\beta\|^2 \, I\!E\left\{\|Z_1\|\,\|X_1\|\right\} \leq \|\beta\|^2 \, I\!E\left\{\|Z_1\|\,\|X_1\|\right\}^q < \infty,$$

we have

$$\limsup_{\|\beta\|\to 0} \left|\ \beta^T I\!E\left[Z_1 X_1^T I\{\beta^T Z_1 X_1^T \beta < 0\}\right] \beta\right| = 0. \tag{24}$$

In other words, τ_λ can be done arbitrary small (just selecting $\lambda \in R^+$ so that $\|\lambda\|$ is small). It says that if $w(b) \equiv 1$, there is $b \in (0,1) > \gamma_{\lambda,a}$ (even for any $a > 0$). It means that (21), (22), (23) and (24) indicate that (41) can be always fulfilled but we may have restricted possibility to depress the influence of "bad" observations.

C4 *The vector equation*

$$\beta^T I\!E\left[w\left(F_\beta(|r_1(\beta)|)\right) Z_1 \left(e_1 - X_1^T \beta\right)\right] = 0 \tag{25}$$

in the variable $\beta \in R^p$ has unique solution $\beta^0 = 0$.

Lemma 1 *Let the conditions* **C1**, **C2**, **C3** *and* **C4** *be fulfilled. Then any sequence $\left\{\hat\beta^{(IWV,n,w)}\right\}_{n=1}^{\infty}$ of the solutions of normal equations $I\!N E_{Z,n}(\hat\beta^{(IWV,n,w)}) = 0$ (see (10)) is weakly consistent.*

For the proof see Víšek (2006b).

$\sqrt{n}$-CONSISTENCY OF THE INSTRUMENTAL WEIGHTED VARIABLES

We will need to enlarge the previous conditions.

NC1 *The density $f_{e|X}(r|X_1 = x)$ is uniformly with respect to x Lipschitz of the first order (with the corresponding constant equal to B_e). Moreover, $f_e'(r)$ exists and is bounded in absolute value by U_e'.*

NC2 *The derivative $w'(\alpha)$ of the weight function is Lipschitz of the first order (with the corresponding constant J_w).*

Lemma 2 *Let the conditions* **C1**, **C2**, **C3**, **C4**, **NC1** *and* **NC2** *be fulfilled. Then any sequence* $\left\{\hat{\beta}^{(IWV,n,w)}\right\}_{n=1}^{\infty}$ *of the solutions of normal equations (10) (or (13))* $I\!N E_{Z,n}(\hat{\beta}^{(IWV,n,w)}) = 0$ *is* $\sqrt{n}$*-consistent.*

Proof:

Throughout the proof for any $r, s \in R$ we shall denote by $[r,s]_{ord} = [\min\{r,s\}, \max\{r,s\}]$ and the same will be true for any other type of intervals, i. e. $(r,s)_{ord}$, $(r,s]_{ord}$ and $[r,s)_{ord}$.

Let us recall that $\hat{\beta}^{(IWV,n,w)}$ is given as solution of (13), i. e. as solution of the equation

$$\sum_{i=1}^{n} w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) Z_i \left(Y_i - X_i'\beta\right) = 0.$$

Rewriting it, we obtain

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) Z_i e_i = \frac{1}{n} \sum_{i=1}^{n} w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) Z_i X_i' \cdot \sqrt{n}\left(\beta - \beta^0\right). \tag{26}$$

Since w' is bounded from below by $-L_w$, we have

$$\sup_{\beta\in R^p} \left| w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) - w\left(F_{\beta}(|r_i(\beta)|)\right)\right| \le L_w \cdot \sup_{v\in R^+} \sup_{\beta\in R^p} \left|F_{\beta}^{(n)}(v) - F_{\beta}(v)\right|.$$

Then according to Lemma A.1

$$\frac{1}{\sqrt{n}} \sup_{\beta\in R^p} \left\| \sum_{i=1}^{n} \left[w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) - w\left(F_{\beta}(|r_i(\beta)|)\right)\right] Z_i e_i \right\|$$

$$\le \frac{1}{\sqrt{n}} \sup_{\beta\in R^p} \sum_{i=1}^{n} \left| w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) - w\left(F_{\beta}(|r_i(\beta)|)\right)\right| \cdot \|Z_i\| \cdot |e_i|$$

$$\le \sqrt{n} \cdot L_w \cdot \sup_{v\in R^+} \sup_{\beta\in R^p} \left|F_{\beta}^{(n)}(v) - F_{\beta}(v)\right| \cdot \frac{1}{n} \sum_{i=1}^{n} \|Z_i\| \cdot |e_i| = \mathcal{O}_p(1)$$

aa $n \to \infty$. Hence (denoting $X = (X_1, X_2, ..., X_n)'$, $Z = (Z_1, Z_2, ..., Z_n)'$ and $e = (e_1, e_2, ..., e_n)'$)

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) Z_i e_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_{\beta}(|r_i(\beta)|)\right) Z_i e_i + R_n^{(1)}(\beta, X, Z, e) \tag{27}$$

where

$$\sup_{\beta\in R^p} \left\| R_n^{(1)}(\beta, X, Z, e)\right\| = \mathcal{O}_p(1)$$

and $\mathcal{O}_p(1)$ is to be understood in the sense that

$$\forall(\varepsilon > 0) \;\; \exists(K_\varepsilon < \infty) \quad \inf_{n\in N} P\left(\left\{\omega \in \Omega \; : \; \sup_{\beta\in R^p} \left\| R_n^{(1)}(\beta, X, Z, e)\right\| < K_\varepsilon \right\}\right) > 1 - \varepsilon. \tag{28}$$

Notice please, that to keep equality in (27), $R_n^{(1)}(\beta, X, Z, e)$ does have to depend on β, X, Z, e and on n. Similarly

$$\frac{1}{n} \sup_{\beta\in R^p} \left\| \sum_{i=1}^{n} \left[w\left(F_{\beta}^{(n)}(|r_i(\beta)|)\right) - w\left(F_{\beta}(|r_i(\beta)|)\right)\right] Z_i X_i' \right\|$$

$$\leq \frac{1}{n} \sup_{\beta \in R^p} \sum_{i=1}^{n} \left| w\left(F_\beta^{(n)}(|r_i(\beta)|)\right) - w\left(F_\beta(|r_i(\beta)|)\right)\right| \cdot \|Z_i\| \cdot \|X_i\|$$

$$\leq \frac{1}{\sqrt{n}} \left\{ L_w \cdot \sup_{v \in R^+} \sup_{\beta \in R^p} \sqrt{n} \left| F_\beta^{(n)}(v) - F_\beta(v) \right| \cdot \frac{1}{n} \sum_{i=1}^{n} \|Z_i\| \cdot \|X_i\| \right\} = o_p(1)$$

as $n \to \infty$. Hence

$$\frac{1}{n} \sum_{i=1}^{n} w\left(F_\beta^{(n)}(|r_i(\beta)|)\right) Z_i X_i' = \frac{1}{n} \sum_{i=1}^{n} w\left(F_\beta(|r_i(\beta)|)\right) Z_i X_i' + R_n^{(2)}(\beta, X, Z, e) \tag{29}$$

where

$$\sup_{\beta \in R^p} \left\| R_n^{(2)}(\beta, X, Z, e) \right\| = o_p(1)$$

and $o_p(1)$ is to be understood in the sense that

$$\forall(\varepsilon > 0, \delta > 0) \quad \exists(n_0 \in N) \quad \forall(n > n_0)$$

$$P\left(\left\{ \omega \in \Omega \ : \ \sup_{\beta \in R^p} \left\| R_n^{(2)}(\beta, X, Z, e) \right\| < \delta \right\}\right) > 1 - \varepsilon. \tag{30}$$

Notice please, that again to keep equality in (29), $R_n^{(2)}(\beta, X, Z, e)$ does have to depend on β, X, Z, e and n. Finally, (26), (27) and (29) gives

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} w\left(F_\beta(|r_i(\beta)|)\right) Z_i e_i + R_n^{(1)}(\beta, X, Z, e)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[w\left(F_\beta(|r_i(\beta)|)\right) Z_i X_i' + R_n^{(2)}(\beta, X, Z, e) \right] \cdot \sqrt{n}\left(\beta - \beta^0\right). \tag{31}$$

Further, let us make some preparatory considerations. Let us recall that by **C1**

$$F_\beta(v) = P\left(\left| e_1 - X_1'\left(\beta - \beta^0\right)\right| < v\right) = \int_{\{|r - x'(\beta - \beta^0)| < v\}} f_{X,e}(x, r) \mathrm{d}x \ \mathrm{d}r$$

$$= \int_{-\infty}^{\infty} \left[\int_{-v + x'(\beta - \beta^0)}^{v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r \right] f_X(x) \mathrm{d}x.$$

Now, for any $\beta \in R^p$ we have

$$F_\beta(v) - F_{\beta^0}(v)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-v + x'(\beta - \beta^0)}^{v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r \right] f_X(x) \mathrm{d}x - \int_{-\infty}^{\infty} \left[\int_{-v}^{v} f_{e|X}(r | X_1 = x) \mathrm{d}r \right] f_X(x) \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left[\int_{-v + x'(\beta - \beta^0)}^{v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r - \int_{-v}^{v} f_{e|X}(r | X_1 = x) \mathrm{d}r \right] f_X(x) \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left[\int_{-v}^{-v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r - \int_{v}^{v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r \right] f_X(x) \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \int_{-v}^{-v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r f_X(x) \mathrm{d}x - \int_{-\infty}^{\infty} \int_{v}^{v + x'(\beta - \beta^0)} f_{e|X}(r | X_1 = x) \mathrm{d}r f_X(x) \mathrm{d}x \tag{32}$$

(where the lower and upper bounds of the integrals should be changed if necessary). Now let us consider the first term of (32). It can be written as

$$\int_{-\infty}^{\infty}\left[\int_{-v}^{-v+x'\left(\beta-\beta^0\right)} f_{e|X}(-v|X_1=x)\mathrm{d}r\right] f_X(x)\mathrm{d}x \tag{33}$$

$$+\int_{-\infty}^{\infty}\left\{\int_{-v}^{-v+x'\left(\beta-\beta^0\right)} \left[f_{e|X}(r|X_1=x)-f_{e|X}(-v|X_1=x)\right]\mathrm{d}r\right\} f_X(x)\mathrm{d}x. \tag{34}$$

Now for $r\in\left[-v,-v+x'\left(\beta-\beta^0\right)\right]_{ord}$

$$\left|f_{e|X}(r|X_1=x)-f_{e|X}(-v|X_1=x)\right|\leq B_e\cdot\left|x'\left[\beta-\beta^0\right]\right|$$

where B_e is given in **NC1**. Then we have for (34) the bounds

$$\left|\int_{-\infty}^{\infty}\int_{-v}^{-v+x'\left(\beta-\beta^0\right)}\left[f_{e|X}(r|X_1=x)-f_{e|X}(-v|X_1=x)\right]\mathrm{d}r f_X(x)\mathrm{d}x\right|$$

$$\leq B_e\cdot\int_{-\infty}^{\infty}\left|x'\left[\beta-\beta^0\right]\right|\cdot\left|\int_{-v}^{-v+x'\left(\beta-\beta^0\right)}\mathrm{d}r\right| f_X(x)\mathrm{d}x = B_e\cdot\int_{-\infty}^{\infty}\left[x'\left(\beta-\beta^0\right)\right]^2 f_X(x)\mathrm{d}x$$

$$\leq B_e\cdot I\!E_{X_1}\|X_1\|^2\cdot\left\|\beta-\beta^0\right\|^2. \tag{35}$$

Notice that the upper bound does not depend on v, i. e. the inequality holds for all $v\in R^+$ (for $v\in R^-$ we have $F_\beta(v)=0$ for any $\beta\in R^p$). Moreover for (33) it holds

$$\int_{-\infty}^{\infty}\left[\int_{-v}^{-v+x'\left(\beta-\beta^0\right)} f_{e|X}(-v|X_1=x)\mathrm{d}r\right] f_X(x)\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}\left[f_{e|X}(-v|X_1=x)\int_{-v}^{-v+x'\left(\beta-\beta^0\right)}\mathrm{d}r\right] f_X(x)\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}\left[f_{e|X}(-v|X_1=x)x'\left(\beta-\beta^0\right)\right] f_X(x)\mathrm{d}x = I\!E_{X_1}\left\{f_{e|X}(-v|X_1)X_1'\right\}\left[\beta-\beta^0\right]. \tag{36}$$

Deriving analogical inequalities as (35) and (36) for the second term of (32), i. e. analogies for

$$\int_{-\infty}^{\infty}\int_{v}^{v+x'\left(\beta-\beta^0\right)}\left[f_{e|X}(r|X_1=x)-f_{e|X}(v|X_1=x)\right]\mathrm{d}r f_X(x)\mathrm{d}x$$

and for

$$\int_{-\infty}^{\infty}\left[\int_{v}^{v+x'\left(\beta-\beta^0\right)} f_{e|X}(v|X_1=x)\mathrm{d}r\right] f_X(x)\mathrm{d}x,$$

we arrive at

$$\sup_{v\in R^+}\left|F_\beta(v)-F_{\beta^0}(v)-\left[I\!E_{X_1}\left\{f_{e|X}(-v|X_1)X_1'\right\}\right.\right.$$

$$\left.\left.-I\!E_{X_1}\left\{f_{e|X}(v|X_1)X_1'\right\}\right]\left[\beta-\beta^0\right]\right|$$

$$\leq 2B_e \cdot I\!E_{X_1} \|X_1\|^2 \cdot \left\|\beta - \beta^0\right\|^2 = \mathcal{O}(\left\|\beta - \beta^0\right\|^2) \text{ as } \beta \to \beta^0. \tag{37}$$

The last inequality also implies that

$$\sup_{v \in R^+} \left| F_\beta(v) - F_{\beta^0}(v) \right| = \mathcal{O}(\left\|\beta - \beta^0\right\|) \tag{38}$$

in this case in the sense

$$\exists (K < \infty) \qquad \sup_{\beta \in R^p} \sup_{v \in R^+} \frac{\left| F_\beta(v) - F_{\beta^0}(v) \right|}{\|\beta - \beta^0\|} < K \tag{39}$$

(keep in mind that for $v \leq 0 \quad F_\beta(v) = F_{\beta^0}(v) = 0$). Now, let us modify (31) as follows

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[w\left(F_\beta(|r_i(\beta)|)\right) - w\left(F_{\beta^0}(|r_i(\beta)|)\right) \right] Z_i e_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n w\left(F_{\beta^0}(|r_i(\beta)|)\right) Z_i e_i + R_n^{(1)}(\beta, X, Z, e)$$

$$= \frac{1}{n} \sum_{i=1}^n \left[w\left(F_\beta(|r_i(\beta)|)\right) - w\left(F_{\beta^0}(|r_i(\beta)|)\right) \right] \cdot Z_i X_i' \cdot \sqrt{n}\left(\beta - \beta^0\right)$$

$$+ \left[\frac{1}{n} \sum_{i=1}^n w\left(F_{\beta^0}(|r_i(\beta)|)\right) Z_i X_i' + R_n^{(2)}(\beta, X, Z, e) \right] \cdot \sqrt{n}\left(\beta - \beta^0\right). \tag{40}$$

To be able to treat the terms in (40) let us consider

$$w\left(F_\beta(|r_i(\beta)|)\right) - w\left(F_{\beta^0}(|r_i(\beta)|)\right) = w'(\xi_i)\left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right]$$

$$= \left[w'(\xi_i) - w'(F_{\beta^0}(|r_i(\beta)|)\right] \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right]$$

$$+ w'(F_{\beta^0}(|r_i(\beta)|) \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right] \tag{41}$$

where $\xi_i \in \left[F_\beta(|r_i(\beta)|), F_{\beta^0}(|r_i(\beta)|)\right]_{ord}$. Moreover, using J_w from **NC2**

$$\left|w'(\xi_i) - w'(F_{\beta^0}(|r_i(\beta)|)\right| \cdot \left|F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right|$$

$$\leq J_w \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right]^2$$

$$\leq J_w \cdot \sup_{v \in R^+} \left[F_\beta(v) - F_{\beta^0}(v)\right]^2 = \mathcal{O}(\left\|\beta - \beta^0\right\|^2) \tag{42}$$

where the last equality is due to (38). Notice that, although the left-hand side of (42) is random, the last but one expression - $J_w \cdot \sup_{v \in R^+} \left[F_\beta(v) - F_{\beta^0}(v)\right]^2$ is not random. Hence the upper bound in (42) holds almost surely. It means that, taking into account (41) and (42), (40) can be rewritten as

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ w'(F_{\beta^0}(|r_i(\beta)|)) \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right] + R_{ni}^{(3)}(\beta, X, Z, e) \right\} Z_i e_i \tag{43}$$

$$+ \frac{1}{\sqrt{n}} \sum_{i=1}^n w\left(F_{\beta^0}(|r_i(\beta)|)\right) Z_i e_i + R_n^{(1)}(\beta, X, Z, e) \tag{44}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\Big\{w'(F_{\beta^0}(|r_i(\beta)|))\cdot\Big[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\Big]$$

$$+R_{ni}^{(4)}(\beta,X,Z,e)\Big\}\cdot Z_iX_i'\cdot\sqrt{n}\left(\beta-\beta^0\right) \qquad (45)$$

$$+\left[\frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}(|r_i(\beta)|)\right) Z_iX_i' + R_n^{(2)}(\beta,X,Z,e)\right]\cdot\sqrt{n}\left(\beta-\beta^0\right) \qquad (46)$$

where

$$\sup_{\beta\in R^p}\left|R_{ni}^{(3)}(\beta,X,Z,e)\right| = \mathcal{O}(\left\|\beta-\beta^0\right\|^2) \quad \text{and} \quad \sup_{\beta\in R^p}\left|R_{ni}^{(4)}(\beta,X,Z,e)\right| = \mathcal{O}(\left\|\beta-\beta^0\right\|^2). \qquad (47)$$

Here the previous two expressions $\mathcal{O}(\|\beta-\beta^0\|^2)$ mean that

$$\exists(\tilde{K}<\infty) \quad \sup_{n\in N}\sup_{i\in N}\sup_{\beta\in R^p}\frac{\left|R_{ni}^{(k)}(\beta,X,Z,e)\right|}{\|\beta-\beta^0\|^2} < \tilde{K} \qquad k=3,4 \quad a.s. \qquad (48)$$

although $R_{ni}^{(k)}(\beta,X,Z,e)$ are random variables (see again (42) and the comments which follow). Let us consider (43), at first the "second term", i. e. $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} R_{ni}^{(3)}(\beta,X,Z,e)Z_ie_i$. We have

$$\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} R_{ni}^{(3)}(\beta,X,Z,e)Z_ie_i\right\| = \sqrt{n}\left\|\beta-\beta^0\right\|\cdot\mathcal{O}_p(\left\|\beta-\beta^0\right\|)\frac{1}{n}\sum_{i=1}^{n}\|Z_i\|\cdot|e_i|$$

$$= \sqrt{n}\left\|\beta-\beta^0\right\|\cdot\mathcal{O}_p(\left\|\beta-\beta^0\right\|).$$

The same is true about the "second term" in (45), since

$$\frac{1}{n}\sum_{i=1}^{n} R_{ni}^{(4)}(\beta,X,Z,e)Z_iX_i' \le \tilde{K}\cdot\left\|\beta-\beta^0\right\|\frac{1}{n}\sum_{i=1}^{n}\|Z_i\|\cdot\|X_i\| = \mathcal{O}_p(\left\|\beta-\beta^0\right\|).$$

So, the relations given between (43) and (46) can be modified to

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w'(F_{\beta^0}(|r_i(\beta)|))\cdot\Big[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\Big]\cdot Z_ie_i \qquad (49)$$

$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w\left(F_{\beta^0}(|r_i(\beta)|)\right)\cdot Z_ie_i + R_n^{(1)}(\beta,X,Z,e) \qquad (50)$$

$$= \frac{1}{n}\sum_{i=1}^{n} w'(F_{\beta^0}(|r_i(\beta)|))\cdot\Big[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\Big]\cdot Z_iX_i'\cdot\sqrt{n}\left(\beta-\beta^0\right) \qquad (51)$$

$$+\left[\frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}(|r_i(\beta)|)\right) Z_iX_i' + R_n^{(2)}(\beta,X,Z,e) + R_n^{(5)}(\beta,X,Z,e)\right]\cdot\sqrt{n}\left(\beta-\beta^0\right) \qquad (52)$$

where for $R_n^{(2)}(\beta,X,Z,e)$ see (29) and (30) and again

$$\sup_{\beta\in R^p}\frac{\left\|R_n^{(5)}(\beta,X,Z,e)\right\|}{\|\beta-\beta^0\|} = \mathcal{O}_p(1) \qquad (53)$$

in the sense of (48). Now, we are going to study (49), (50), (51) and (52) one by one. Recalling that, according to (11), $F_{\beta^0}(v) = P(|Y_1 - X_1'\beta^0| < v) = P(|e_1| < v) = P(-v < e_1 < v)$, for any pair $v_1, v_2 \in R$, assuming that $0 \le v_1 < v_2$, we have

$$F_{\beta^0}(v_2) - F_{\beta^0}(v_1) = P(|e_1| < v_2) - P(|e_1| < v_1) = P(-v_2 < e_1 \le -v_1) + P(v_1 \le e_1 < v_2)$$
$$\le 2 \cdot B_e \cdot |v_1 - v_2| \tag{54}$$

(for B_e see **NC1**), so that

$$\left|F_{\beta^0}(r_i(\beta)) - F_{\beta^0}(r_i(\beta^0))\right| \le 2 \cdot B_e \cdot \|X_i\| \cdot \left\|\beta - \beta^0\right\|$$

and, due to **NC2** and the fact that $\Big||a| - |b|\Big| \le |a - b|$,

$$\left|w'(F_{\beta^0}(|r_i(\beta)|)) - w'(F_{\beta^0}(|r_i(\beta^0)|))\right| \le J_w \cdot B_e \cdot \|X_i\| \cdot \left\|\beta - \beta^0\right\|. \tag{55}$$

It means that, employing also (38),

$$\left|w'(F_{\beta^0}(|r_i(\beta)|)) - w'(F_{\beta^0}(|r_i(\beta^0)|))\right| \cdot \left|F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right|$$
$$\le \left|w'(F_{\beta^0}(|r_i(\beta)|)) - w'(F_{\beta^0}(|r_i(\beta^0)|))\right| \cdot \sup_{v \in R^+} \left|F_\beta(v) - F_{\beta^0}(v)\right|$$
$$\le J_w \cdot B_e \cdot \|X_i\| \cdot \left\|\beta - \beta^0\right\| \cdot \sup_{v \in R^+} \left|F_\beta(v) - F_{\beta^0}(v)\right| = \|X_i\| \cdot \mathcal{O}(\left\|\beta - \beta^0\right\|^2). \tag{56}$$

Let us again repeat that, denoting

$$R_n^{(6)}(\beta) = J_w \cdot B_e \cdot \left\|\beta - \beta^0\right\| \cdot \sup_{v \in R^+} \left|F_\beta(v) - F_{\beta^0}(v)\right|,$$

the last equality in (56) means that:

$$\exists (K < \infty) \quad \sup_{n \in N} \sup_{\beta \in R^p} \frac{\left|R_n^{(6)}(\beta)\right|}{\|\beta - \beta^0\|^2} < K.$$

Finally,

$$w'(F_{\beta^0}(|r_i(\beta)|)) \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta^0)|)\right]$$
$$= w'(F_{\beta^0}(|r_i(\beta^0)|)) \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right] + \|X_i\| \cdot R_n^{(7)}(\beta)$$

where $\left|R_n^{(7)}(\beta)\right| \le \left|R_n^{(6)}(\beta)\right|$ for any $\beta \in R^p$, i. e.

$$\sup_{\beta \in R^p} \frac{\left|R_n^{(7)}(\beta)\right|}{\|\beta - \beta^0\|^2} = \mathcal{O}(1), \tag{57}$$

again in the sense described in (48). Hence (49) can be written as

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{w'(F_{\beta^0}(|r_i(\beta^0)|)) \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right] + \|X_i\| \cdot R_n^{(7)}(\beta)\right\} \cdot Z_i e_i.$$

As

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\|X_i\|\cdot\|Z_i\|\cdot|e_i|\cdot R_n^{(7)}(\beta)=\frac{1}{n}\sum_{i=1}^{n}\|X_i\|\cdot\|Z_i\|\cdot|e_i|\cdot\sqrt{n}\left\|\beta-\beta^0\right\|\cdot\frac{R_n^{(7)}(\beta)}{\|\beta-\beta^0\|},$$

taking into account (57), we can finally write (49) as

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{w'(F_{\beta^0}(|e_i|))\cdot\left[F_\beta(|r_i(\beta)|)-F_{\beta^0}(|r_i(\beta)|)\right]\right\}\cdot Z_i\cdot e_i+\sqrt{n}\left(\beta-\beta^0\right)\cdot R_n^{(8)}(\beta,X,Z,e) \quad (58)$$

where

$$\sup_{\beta\in R^p}\frac{\left|R_n^{(8)}(\beta,X,Z,e)\right|}{\|\beta-\beta^0\|}=\mathcal{O}_p(1),$$

of course again in the sense described in (48). Now, recalling that

$$I\!E_{X_1}\left\{f_{e|X}(v|X_1)X_1\right\}=\int_{-\infty}^{\infty}f_{e|X}(v|X_1=x)xf_X(x)\mathrm{d}x$$

and the fact that $f_{e|X}(v|X_1=x)$ is Lipschitz (with the corresponding constant B_e, see **NC1**), we easy verify that

$$\begin{aligned}&\left\|I\!E_{X_1}\left\{f_{e|X}(v_1|X_1)X_1\right\}-I\!E_{X_1}\left\{f_{e|X}(v_2|X_1)X_1\right\}\right\|\\&=\left\|\int_{-\infty}^{\infty}\left[f_{e|X}(v_1|X_1=x)-f_{e|X}(v_2|X_1=x)\right]xf_X(x)\mathrm{d}x\right\|\\&\le B_e\cdot|v_1-v_2|\left\|\int_{-\infty}^{\infty}xf_X(x)\mathrm{d}x\right\|=B_e\cdot|v_1-v_2|\cdot I\!E_{X_1}\|X_1\|\end{aligned}$$

and hence

$$\begin{aligned}&\left|\left[I\!E_{X_1}\left\{f_{e|X}(r_i(\beta)|X_1)X_1'\right\}-I\!E_{X_1}\left\{f_{e|X}(r_i(\beta^0)|X_1)X_1'\right\}\right]\cdot\left[\beta-\beta^0\right]\right|\\&\le B_e\cdot I\!E_{X_1}\left\{\left|r_i(\beta)-r_i(\beta^0)\right|\cdot\|X_1\|\right\}\cdot\left\|\beta-\beta^0\right\|\le B_e\cdot I\!E_{X_1}\|X_1\|^2\cdot\left\|\beta-\beta^0\right\|^2.\end{aligned}$$

Together with (37) the last equality implies that

$$\begin{aligned}&\Big|F_\beta(r_i(\beta))-F_{\beta^0}(r_i(\beta))-\Big[I\!E_{X_1}\left\{f_{e|X}(-e_i|X_1)X_1'\right\}\\&\qquad -I\!E_{X_1}\left\{f_{e|X}(e_i|X_1)X_1'\right\}\Big]\left[\beta-\beta^0\right]\Big|\\&\le 4\cdot B_e\max\left\{I\!E_{X_1}\|X_1\|^2,1\right\}\cdot\left\|\beta-\beta^0\right\|^2. \quad (59)\end{aligned}$$

So, we found that (49) is equivalent to

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}w'(F_{\beta^0}(|e_i|))\cdot\left[E_{X_1}\left\{f_{e|X}(-e_i|X_1)X_1'\right\}-I\!E_{X_1}\left\{f_{e|X}(e_i|X_1)X_1'\right\}\right]\cdot\left[\beta-\beta^0\right]\cdot Z_i\cdot e_i\\&\qquad +R_n^{(9)}(\beta,X,Z,e)\sqrt{n}\left[\beta-\beta^0\right] \quad (60)\end{aligned}$$

where again

$$\sup_{\beta \in R^p} \frac{\left| R_n^{(9)}(\beta, X, Z, e) \right|}{\|\beta - \beta^0\|} = \mathcal{O}_p(1),$$

in the sense of (48). That concludes the considerations about (49).

Let us turn to (50). Recalling that $r_i(\beta) - r_i(\beta^0) = X_i'(\beta - \beta^0)$, $r_i(\beta^0) = e_i$ and that

$$F_{\beta^0}(v) = F_e(v) - F_e(-v),$$

we have

$$\begin{aligned} &F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta^0)|) \\ &= F_e(r_i(\beta)) - F_e(-r_i(\beta)) - F_e(r_i(\beta^0))) + F_e(-r_i(\beta^0)) \\ &= \Big(f_e(r_i(\beta^0)) - f_e(-r_i(\beta^0))\Big) \cdot \Big(r_i(\beta) - r_i(\beta^0)\Big) + \frac{1}{2} f_e'(\theta_i) \cdot \Big(r_i(\beta) - r_i(\beta^0)\Big)^2 \\ &= (f_e(e_i) - f_e(-e_i)) \cdot X_i' \left(\beta - \beta^0\right) + \frac{1}{2} f_e'(\theta_i) \cdot \left[X_i' \left(\beta - \beta^0\right)\right]^2 \end{aligned}$$

where θ_i is an appropriate point from $\left[r_i(\beta), r_i(\beta^0)\right]_{ord}$. Since $|f_e'(v)|$ is bounded by U_e' (see **NC1**), we have

$$\left| F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|e_i|) - \Big(f_e(e_i) - f_e(-e_i)\Big) \cdot X_i' \left(\beta - \beta^0\right) \right| \le U_e' \cdot \|X_i\|^2 \cdot \left\|\beta - \beta^0\right\|^2 \qquad (61)$$

and also

$$\Big| F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|e_i|)) \Big| \le U_e \cdot \|X_i\| \cdot \left\|\beta - \beta^0\right\| \qquad (62)$$

(for U_e see **C1** and the remark below **C1**). Then

$$\begin{aligned} &\left[w\left(F_{\beta^0}(|r_i(\beta)|)\right) - w\left(F_{\beta^0}(|r_i(\beta^0)|)\right)\right] \cdot Z_i e_i = w'(\xi_i)\left(F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta^0)|)\right) \cdot Z_i e_i \\ &= \left[w'(\xi_i) - w'(F_{\beta^0}(|r_i(\beta^0)|))\right] \cdot \left(F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta^0)|)\right) \cdot Z_i e_i \\ &\quad + w'(F_{\beta^0}(|r_i(\beta^0)|))\left(F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta^0)|)\right) \cdot Z_i e_i \end{aligned}$$

where ξ_i is again an appropriate point from $\left[F_{\beta^0}(|r_i(\beta)|), F_{\beta^0}(|r_i(\beta^0)|)\right]_{ord}$. Due to (55) and (62), we have

$$\left|w'(\xi_i) - w'(F_{\beta^0}(|r_i(\beta^0)|))\right| \cdot \left|F_{\beta^0}(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta^0)|)\right| = J_w \cdot B_e \cdot U_e \cdot \|X_i\|^2 \cdot \left\|\beta - \beta^0\right\|^2$$

and hence, due to (61),

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[w\left(F_{\beta^0}(|r_i(\beta)|)\right) - w\left(F_{\beta^0}(|r_i(\beta^0)|)\right)\right] \cdot Z_i e_i \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n w'(F_{\beta^0}(|e_i|)) \left[\Big(f_e(e_i) - f_e(-e_i)\Big) \cdot X_i' \left(\beta - \beta^0\right)\right] \cdot Z_i e_i + R_n^{(10)}(\beta, X, Z, e) \sqrt{n} \left(\beta - \beta^0\right) \end{aligned} \qquad (63)$$

with $\sup_{\beta\in R^p}\left|R_n^{(10)}(\beta,X,Z,e)\right| = \mathcal{O}_p\left(\|\beta-\beta^0\|\right)$ again with the sense described in previous. It implies that (50) can be written as

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n\left\{w\left(F_{\beta^0}(|e_i|)\right)+w'(F_{\beta^0}(|e_i|))\left[\left(f_e(e_i)-f_e(-e_i)\right)\cdot X_i'\left(\beta-\beta^0\right)\right]\right\}\cdot Z_ie_i$$
$$+R_n^{(10)}(\beta,X,Z,e)\sqrt{n}\left(\beta-\beta^0\right). \tag{64}$$

So, we may conclude the considerations about (49) and (50) and to write them as the sum of three terms, namely

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n w\left(F_{\beta^0}(|e_i|)\right)\cdot Z_ie_i, \tag{65}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n w'(F_{\beta^0}(|e_i|))\left\{\left[\left(f_e(e_i)-f_e(-e_i)\right)\cdot X_i'\right]\right.$$
$$\left.+\left[E_{X_1}\left\{f_{e|X}(-e_i|X_1)X_1'\right\}-I\!E_{X_1}\left\{f_{e|X}(e_i|X_1)X_1'\right\}\right]\right\}\left[\beta-\beta^0\right]\cdot Z_ie_i \tag{66}$$

and

$$R_n^{(9)}(\beta,X,Z,e)+R_n^{(10)}(\beta,X,Z,e) \tag{67}$$

where

$$\sup_{\beta\in R^p}\frac{\left|R_n^{(9)}(\beta,X,Z,e)\right|+\left|R_n^{(10)}(\beta,X,Z,e)\right|}{\|\beta-\beta^0\|}=\mathcal{O}_p(1)$$

in the sense of (48). Moreover, (66) can be written as follows.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n w'(F_{\beta^0}(|e_i|))\left\{\left[f_e(e_i)-I\!E_{X_1}\left\{f_{e|X}(e_i|X_1)X_1'\right\}\right]\right.$$
$$\left.-\left[f_e(-e_i)-I\!E_{X_1}\left\{f_{e|X}(-e_i|X_1)X_1'\right\}\right]\right\}\cdot\left(\beta-\beta^0\right)\cdot Z_ie_i. \tag{68}$$

Notice that due to CLT, (65) is $\mathcal{O}_p(1)$. Further, let us recall that under the assumptions of the lemma, $\hat\beta^{(IWV,n,w)}$ is consistent, i. e. $\left\|\hat\beta^{(IWS,n,w)}-\beta^0\right\| = o_p(1)$, see Lemma 1. Then plugging $\hat\beta^{(IWS,n,w)}$ into (67) and (68), we find that both expressions are $o_p(1)$. Finally we conclude that when plugging in left hand side of normal equations $\hat\beta^{(IWV,n,w)}$, we get $\mathcal{O}_p(1)$.

Let us continue with (51). We have

$$\frac{1}{n}\sum_{i=1}^n w'(F_{\beta^0}(|r_i(\beta)|))\cdot\left[F_\beta(|r_i(\beta)|)-F_{\beta^0}(|r_i(\beta)|)\right]$$
$$=\frac{1}{n}\sum_{i=1}^n\left[w'(F_{\beta^0}(|r_i(\beta)|))-w'(F_{\beta^0}(|e_i|))\right]\cdot\left[F_\beta(|r_i(\beta)|)-F_{\beta^0}(|r_i(\beta)|)\right]$$
$$+\frac{1}{n}\sum_{i=1}^n w'(F_{\beta^0}(|e_i|))\cdot\left[F_\beta(|r_i(\beta)|)-F_{\beta^0}(|r_i(\beta)|)\right]$$

and since, due to (38), (55) and due to existence of $I\!E\|X_1\|$,

$$\frac{1}{n}\sum_{i=1}^{n}\left|w'(F_{\beta^0}(|r_i(\beta)|)) - w'(F_{\beta^0}(|e_i|))\right| \cdot \left|F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right|$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\left|w'(F_{\beta^0}(|r_i(\beta)|)) - w'(F_{\beta^0}(|e_i|))\right| \cdot \sup_{v\in R^+}\left|F_\beta(v) - F_{\beta^0}(v)\right|$$

$$\leq J_w \cdot B_e \cdot \left\|\beta - \beta^0\right\| \cdot \sup_{v\in R^+}\left|F_\beta(v) - F_{\beta^0}(v)\right| \cdot \frac{1}{n}\sum_{i=1}^{n}\|X_i\|,$$

we have (for K see (39))

$$\left\|\frac{1}{n}\sum_{i=1}^{n}\left[w'(F_{\beta^0}(|r_i(\beta)|)) - w'(F_{\beta^0}(|e_i|))\right] \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right] \cdot Z_i \cdot X_i'\right\|$$

$$\leq J_w \cdot B_e \cdot K \cdot \left\|\beta - \beta^0\right\| \cdot \sup_{v\in R^+}\left|F_\beta(v) - F_{\beta^0}(v)\right| \cdot \frac{1}{n}\sum_{i=1}^{n}\|X_i\| \cdot \|Z_i\| = \mathcal{O}_p(\left\|\beta - \beta^0\right\|^2)$$

again in a uniform sense described in (48). Hence (51) can be written as

$$\frac{1}{n}\sum_{i=1}^{n} w'(F_{\beta^0}(|e_i|)) \cdot \left[F_\beta(|r_i(\beta)|) - F_{\beta^0}(|r_i(\beta)|)\right] Z_i X_i' + R_n^{(11)}(\beta, X, Z, e) \tag{69}$$

where $\sup_{\beta\in R^p}\left\{\left\|R_n^{(11)}(\beta, X, Z, e)\right\| \cdot \|\beta - \beta^0\|^{-2}\right\} = \mathcal{O}_p(1)$ (again in the previously explained sense). Taking into account (37), we conclude that (51) can be written as

$$\left\{\frac{1}{n}\sum_{i=1}^{n}\left\{w'(F_{\beta^0}(|e_i|)) \cdot \left[I\!E_{X_1}\left\{f_{e|X}(-v|X_1)X_1'\right\}\right.\right.\right.$$

$$\left.\left.\left. - I\!E_{X_1}\left\{f_{e|X}(v|X_1)X_1'\right\}\right] \cdot Z_i X_i'\right\}\left[\beta - \beta^0\right] + R_n^{(12)}(\beta, X, Z, e)\right\} \cdot \sqrt{n}\left[\beta - \beta^0\right] \tag{70}$$

where again $\sup_{\beta\in R^p}\left\{\left\|R_n^{(12)}(\beta, X, Z, e)\right\| \cdot \|\beta - \beta^0\|^{-2}\right\} = \mathcal{O}_p(1)$. It remains to study (52). Along similar lines as in previous we arrive at

$$\left[\frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) Z_i X_i' + R_n^{(13)}(\beta, X, Z, e)\right] \cdot \sqrt{n}\left(\beta - \beta^0\right) \tag{71}$$

where again $\sup_{\beta\in R^p}\left\{\left\|R_n^{(13)}(\beta, X, Z, e)\right\| \cdot \|\beta - \beta^0\|^{-2}\right\} = \mathcal{O}_p(1)$. Now, taking into account (70) and (71), we conclude that (51) and (52) can be given for $\beta = \hat{\beta}^{(IWV,T,w)}$ as

$$\frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) Z_i X_i' \cdot \sqrt{n}\left(\hat{\beta}^{(IWV,T,w)} - \beta^0\right) + R_n^{(14)}(\hat{\beta}^{(IWV,T,w)}, X, Z, e)$$

with $\sup_{\beta\in R^p}\left\|R_n^{(14)}(\hat{\beta}^{(IWV,T,w)}, X, Z, e)\right\| = o_p(1)$. Since

$$\frac{1}{n}\sum_{i=1}^{n} w\left(F_{\beta^0}(|e_i|)\right) Z_i X_i' \tag{72}$$

converges in probability to a regular matrix, taking into account (65), (66), (67), (70) and (72) and employing Lemma A.2, we conclude the proof of the present lemma. □

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Appendix

Lemma A.1 *Let the conditions* **C1** *hold and fix arbitrary $\varepsilon > 0$. Then there is a constant $K < \infty$ and $n_\varepsilon \in N$ so that for all $n > n_\varepsilon$*

$$P\left(\left\{\omega \in \Omega : \sup_{v \in R^+} \sup_{\beta \in R^p} \sqrt{n}\left|F_\beta^{(n)}(v) - F_\beta(v)\right| < K\right\}\right) > 1 - \varepsilon. \qquad \text{(A.73)}$$

For the **proof** of lemma see Víšek (2006a).

Lemma A.2 *Let for some $p \in N$, $\left\{\mathcal{V}^{(n)}\right\}_{n=1}^{\infty}$, $\mathcal{V}^{(n)} = \left\{v_{ij}^{(n)}\right\}_{i=1,2,...,p}^{j=1,2,...,p}$ be a sequence of $(p \times p)$ matrixes such that for $i = 1, 2, ..., p$ and $j = 1, 2, ..., p$*

$$\lim_{n \to \infty} v_{ij}^{(n)} = q_{ij} \qquad \textit{in probability} \qquad \text{(A.74)}$$

where $Q = \{q_{ij}\}_{i=1,2,...,p}^{j=1,2,...,p}$ is a fixed nonrandom regular matrix. Moreover, let $\left\{\theta^{(n)}\right\}_{n=1}^{\infty}$ be a sequence of p–dimensional random vectors such that

$$\exists\ (\varepsilon > 0)\ \forall\ (K > 0)\ \limsup_{n \to \infty} P\left(\|\theta^{(n)}\| > K\right) > \varepsilon.$$

Then

$$\exists\ (\ \delta > 0) \quad \forall\ (H > 0)$$

so that

$$\limsup_{n \to \infty}\ P\left(\left\|\mathcal{V}^{(n)}\theta^{(n)}\right\| > H\right) > \delta.$$

Proof: Due to (A.74) the matrix $\mathcal{V}^{(n)}$ is regular in probability. Let then $0 < \lambda_{1n} < \lambda_{2n} < ... < \lambda_{pn}$ and $z_{1n}, z_{2n}, ..., z_{pn}$ be eigenvalues and corresponding eigenvectors (selected to be mutually orthogonal) of the matrix $[\mathcal{V}^{(n)}]^T\mathcal{V}^{(n)}$. Let us write $\theta^{(n)} = \sum_{j=1}^p a_{jn}z_{jn}$ (for an appropriate vector $a_n = (a_{1n}, a_{1n}, ..., a_{pn})^T$). Then we have

$$\left\|\mathcal{V}^{(n)}\theta^{(n)}\right\|^2 = \sum_{j=1}^p [a_{jn}]^2\lambda_{jn}\|z_{jn}\|^2 \leq \lambda_{1n}\|\theta^{(n)}\|. \qquad \text{(A.75)}$$

Moreover, denoting λ_1 the smallest eigenvalue of the matrix Q^TQ, we have $\lambda_{1n} \to \lambda_1$ in probability as $n \to \infty$. The assertion of the lemma then follows from (A.75), see also Víšek (1996) or (2002a). □

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